Linear independence

Review.

- $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all linear combinations

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m.$$

- $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is a vector space.

Example 1. Is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}\right\}$ equal to $\mathbb{R}^3$?

Solution. The span is equal to $\mathbb{R}^3$ if and only if the system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 1 & 2 & 1 & b_2 \\ 1 & 3 & 3 & b_3 \end{array}\right]$$

is consistent for all b_1, b_2, b_3.

- What went "wrong"? Well, the three vectors in the span satisfy

$$\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Armin Straub
astraub@illinois.edu

- Hence, $\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}=\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$.
- We are going to say that the three vectors are **linearly dependent** because they satisfy

$$-3\begin{bmatrix}1\\1\\1\end{bmatrix}+2\begin{bmatrix}1\\2\\3\end{bmatrix}-\begin{bmatrix}-1\\1\\3\end{bmatrix}=\mathbf{0}.$$

Definition 2. Vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_p$ are said to be **linearly independent** if the equation

$$x_1\boldsymbol{v}_1+x_2\boldsymbol{v}_2+\ldots+x_p\boldsymbol{v}_p=\mathbf{0}$$

has only the trivial solution (namely, $x_1=x_2=\ldots=x_p=0$).

Likewise, $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_p$ are said to be **linearly dependent** if there exist coefficients $x_1,\ldots,x_p$, not all zero, such that

$$x_1\boldsymbol{v}_1+x_2\boldsymbol{v}_2+\ldots+x_p\boldsymbol{v}_p=\mathbf{0}.$$

Example 3.

- Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-1\\1\\3\end{bmatrix}$ independent?
- If possible, find a linear dependence relation among them.

Solution.

Linear independence of matrix columns

- Note that a linear dependence relation, such as

$$3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0},$$

can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \mathbf{0}.$$

- Hence, each linear dependence relation among the columns of a matrix A corresponds to a nontrivial solution to $A\boldsymbol{x} = \mathbf{0}$.

Theorem 4. Let A be an $m \times n$ matrix.

The columns of A are linearly independent.
$\iff$ $A\boldsymbol{x} = \mathbf{0}$ has only the solution $\boldsymbol{x} = 0$.
$\iff$ $\mathrm{Nul}(A) = \{\mathbf{0}\}$
$\iff$ A has n pivots. (one in each column)

Special cases

- A set of a single nonzero vector $\{\boldsymbol{v}_1\}$ is always linearly independent.

  Why?

- A set of two vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly independent if and only if neither of the vectors is a multiple of the other.

  Why?

- A set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ containing the zero vector is linearly dependent.

  Why?

- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. In other words:

  Any set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.

  Why?

Example 5. With the least amount of work possible, decide which of the following sets of vectors are linearly independent.

(a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

(c) columns of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \end{bmatrix}$

(d) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution.

A basis of a vector space

Definition 6. A set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in V is a **basis** of V if

- $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$, and
- the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are linearly independent.

In other words, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in V is a basis of V if and only if every vector $\boldsymbol{w}$ in V can be uniquely expressed as $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_p\boldsymbol{v}_p$.

Example 7. Let $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis of $\mathbb{R}^3$.

It is called the **standard basis**.

Solution.

Definition 8. V is said to have **dimension** p if it has a basis consisting of p vectors.

Armin Straub
astraub@illinois.edu

This definition makes sense because if V has a basis of p vectors, then every basis of V has p vectors. Why? (Think of $V = \mathbb{R}^3$.)

Example 9. $\mathbb{R}^3$ has dimension 3. Likewise, $\mathbb{R}^n$ has dimension n.

Example 10. Not all vector spaces have a finite basis. For instance, the vector space of all polynomials has *infinite dimension*.

Its standard basis is

Recall that vectors in V form a **basis** of V if they span V and if they are linearly independent. If we know the dimension of V, we only need to check one of these two conditions:

Theorem 11. Suppose that V has dimension d.

- A set of d vectors in V are a basis if they span V.
- A set of d vectors in V are a basis if they are linearly independent.

Example 12. Are the following sets a basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

Solution.

Example 13. Let $\mathbb{P}_2$ be the space of polynomials of degree at most 2.

- What is the dimension of $\mathbb{P}_2$?
- Is $\{t, 1-t, 1+t-t^2\}$ a basis of $\mathbb{P}_2$?

Solution. The standard basis for $\mathbb{P}_2$ is

Shrinking and expanding sets of vectors

We can find a basis for $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ by discarding, if necessary, some of the vectors in the spanning set.

Example 14. Produce a basis of $\mathbb{R}^2$ from the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution.

Example 15. Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \text{ real} \right\}.$$

Solution.

Every set of linearly independent vectors can be extended to a basis.

In other words, let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ be linearly independent vectors in V. If V has dimension d, then we can find vectors $\boldsymbol{v}_{p+1}, \ldots, \boldsymbol{v}_d$ such that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ is a basis of V.

Example 16. Consider

$$H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- What is the dimension of this subspace of $\mathbb{R}^3$?
- Give a basis for H, and then extend it to a basis of $\mathbb{R}^3$.

Solution.

Armin Straub
astraub@illinois.edu

Checking our understanding

Example 17. Subspaces of $\mathbb{R}^3$ can have dimension $0, 1, 2, 3$.

- The only 0-dimensional subspace is
- A 1-dimensional subspace
- A 2-dimensional subspace
- The only 3-dimensional subspace is

True or false?

- Suppose that V has dimension n. Then any set in V containing more than n vectors must be linearly dependent.
- The space $\mathbb{P}_n$ of polynomials of degree at most n has dimension $n+1$.
- The vector space of functions $f: \mathbb{R} \to \mathbb{R}$ is infinite-dimensional.
- Consider $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$. If one of the vectors, say $\boldsymbol{v}_k$, in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span V.