LU decomposition

Elementary matrices

Example 1.

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$

Definition 2. An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

> The result of an elementary row operation on A is EA
>
> where E is an elementary matrix (namely, the one obtained by performing the same row operation on the appropriate identity matrix).

Example 3. Elementary matrices are **invertible** because row operations are reversible.

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} =$

  We write $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, but more on inverses soon.
- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} =$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} =$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} =$

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Example 4. Keeping track of the elementary matrices during Gaussian elimination on A:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$$
$$EA = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

Note that:

$$A = E^{-1} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

We factored A as the product of a lower and upper triangular matrix!

We say that A has **triangular factorization**.

$A = LU$ is known as the **LU decomposition** of A.

Definition 5.

lower triangular

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & * & \cdots & * \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}$$

missing entries are 0

Example 6. Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ as $A = LU$.

Solution. We begin with $R2 \to R2 - 2R1$ followed by $R3 \to R3 + R1$:

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$
$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

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$$E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= U$$

The factor L is given by:

$$\begin{aligned} L &= E_1^{-1}E_2^{-1}E_3 \\ &= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \\ &= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{aligned}$$

In conclusion, we found the following LU decomposition of A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = LU = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Once we have $A = LU$, it is simple to solve $A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{aligned} & A\boldsymbol{x} = \boldsymbol{b} \\ \iff & L(U\boldsymbol{x}) = \boldsymbol{b} \\ \iff & L\boldsymbol{c} = \boldsymbol{b} \quad \text{and} \quad U\boldsymbol{x} = \boldsymbol{c} \end{aligned}$$

Both of the final systems are triangular and hence easily solved:

- $L\boldsymbol{c} = \boldsymbol{b}$ by forward substitution to find $\boldsymbol{c}$, and then
- $U\boldsymbol{x} = \boldsymbol{c}$ by backward substitution to find $\boldsymbol{x}$.

Important practical point: can be quickly repeated for many different $\boldsymbol{b}$.

Example 7. Solve $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}$.

Solution.

Triangular factors for any matrix

Can we factor any matrix A as $A = LU$?

Yes, almost! Think about the process of Gaussian elimination.

- In each step, we use a pivot to produce zeros below it.
  The corresponding elementary matrices are lower diagonal!
- The only other thing we might have to do, is a row exchange.
  Namely, if we run into a zero in the position of the pivot.
- All of these row exchanges can be done at the beginning!

Definition 8. A **permutation matrix** is one that is obtained by performing row exchanges on an identity matrix.

Theorem 9. For any matrix A there is a permutation matrix P such that $PA = LU$.

Example 10. Consider $A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$.

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