Introduction to systems of linear equations

These slides are based on Section 1 in *Linear Algebra and its Applications* by David C. Lay.

Definition 1. A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written as

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b.$$

Example 2. Which of the following equations are linear?

- $4x_1 - 5x_2 + 2 = x_1$
- $x_2 = 2(\sqrt{6} - x_1) + x_3$
- $4x_1 - 6x_2 = x_1 x_2$
- $x_2 = 2\sqrt{x_1} - 7$

Definition 3.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.
- A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Example 4. (Two equations in two variables)

In each case, sketch the set of all solutions.

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ -x_1 & + & x_2 & = & 0 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ -4x_1 & - & 2x_2 & = & -2 \end{array}$$

Theorem 5. A linear system has either

- no solution, or
- one unique solution, or
- infinitely many solutions.

Definition 6. A system is **consistent** if a solution exists.

How to solve systems of linear equations

Strategy: replace system with an equivalent system which is easier to solve

Definition 7. Linear systems are **equivalent** if they have the same set of solutions.

Example 8. To solve the first system from the previous example:

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 1 \\ -x_1 & + & x_2 & = & 0 \end{array} \quad \overset{R2 \to R2 + R1}{\rightsquigarrow} \quad \begin{array}{rcrcl} x_1 & + & x_2 & = & 1 \\ & & 2x_2 & = & 1 \end{array}$$

Once in this **triangular** form, we find the solutions by **back-substitution**:

$$x_2 = 1/2, \qquad x_1 = \ldots$$

Example 9. The same approach works for more complicated systems.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \quad \downarrow R3 \to R3 + 4R1$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \downarrow R3 \to R3 + \tfrac{3}{2}R2$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array}$$

By back-substitution:

$$x_3 = 3, \qquad x_2 = \ldots, \qquad x_1 = \ldots$$

It is always a good idea to check our answer. Let us check that $(29, 16, 3)$ indeed solves the original system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \ldots$$

Armin Straub
astraub@illinois.edu

Matrix notation

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$

$$\left[\begin{array}{cc} & \\ & \end{array}\right]$$

(coefficient matrix)

$$\left[\begin{array}{cc|c} & & \\ & & \end{array}\right]$$

(augmented matrix)

Definition 10. An **elementary row operation** is one of the following:

- **(replacement)** Add one row to a multiple of another row.
- **(interchange)** Interchange two rows.
- **(scaling)** Multiply all entries in a row by a nonzero constant.

Definition 11. Two matrices are **row equivalent**, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Theorem 12. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example 13. Here is the previous example in matrix notation.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \qquad \downarrow R3 \to R3 + 4R1$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \qquad \downarrow R3 \to R3 + \tfrac{3}{2}R2$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Instead of back-substitution, we can continue with row operations:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & 2x_2 & & & = & 32 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We again find the solution $(x_1, x_2, x_3) = (29, 16, 3)$.

Armin Straub
astraub@illinois.edu

Row reduction and echelon forms

Definition 14. A matrix is in **echelon form** (or **row echelon form**) if:

(1) Each leading entry (i.e. leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it.

(2) All entries in a column below a leading entry are zero.

(3) All nonzero rows are above any rows of all zeros.

Example 15. Here is a representative matrix in echelon form.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($*$ stands for any value, and $\blacksquare$ for any nonzero value.)

Example 16. Are the following matrices in echelon form?

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} \blacksquare & 0 & 0 \\ * & \blacksquare & 0 \\ * & 0 & \blacksquare \\ * & 0 & 0 \end{bmatrix}$

Definition 17. A leading entry in an echelon form is called a **pivot**.

Definition 18. A matrix is in **reduced echelon form** if, in addition to being in echelon form, it also satisfies:

(4) Each pivot is 1.

(5) Each pivot is the only nonzero entry in its column.

Armin Straub
astraub@illinois.edu

Example 19. Our initial matrix in echelon form put into reduced echelon form:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \blacksquare & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & \blacksquare & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Locate the pivots!

Example 20. Are the following matrices in reduced echelon form?

(a) $\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -2 & 3 & 2 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Theorem 21. (Uniqueness of the reduced echelon form) Each matrix is row equivalent to one and only one reduced echelon matrix.

Question. Is the same statement true for the echelon form?

Example 22. Row reduce to echelon form (often called **Gaussian elimination**) and then to reduced echelon form (often called **Gauss–Jordan elimination**):

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution.

Hence, the reduced echelon form is:

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Solution of linear systems via row reduction

After row reduction to echelon form, we can easily solve a linear system.
(especially after reduction to reduced echelon form)

Example 23.

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \rightsquigarrow \begin{array}{rrrrrcl} x_1 & +6x_2 & & +3x_4 & & = & 0 \\ & & x_3 & -8x_4 & & = & 5 \\ & & & & x_5 & = & 7 \end{array}$$

- The pivots are located in columns $1, 3, 5$. The corresponding variables x_1, x_3, x_5 are called **pivot variables** (or **basic variables**).
- The remaining variables x_2, x_4 are called **free variables**.
- We can solve each equation for the pivot variables in terms of the free variables (if any). Here, we get:

$$\begin{array}{rrrrrcl} x_1 & +6x_2 & & +3x_4 & & = & 0 \\ & & x_3 & -8x_4 & & = & 5 \\ & & & & x_5 & = & 7 \end{array} \qquad \left\{\begin{array}{ll} x_1 & = \\ x_2 & \text{free} \\ x_3 & = \\ x_4 & \text{free} \\ x_5 & = \end{array}\right.$$

- This is the **general solution** of this system. The solution is in parametric form, with parameters given by the free variables.
- Just to make sure: Is the above system consistent? Does it have a unique solution?

Example 24. Find a parametric description of the solution set of:

$$\begin{array}{rrrrrcl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \end{array}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right].$$

We determined earlier that its reduced echelon form is

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

The pivot variables are ...

The free variables are ...

Armin Straub
astraub@illinois.edu

Hence, we find the general solution as:

$$\begin{cases} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{cases}$$

Questions of existence and uniqueness

The question whether a system has a solution and whether it is unique, is easier to answer than to determine the solution set.

All we need is an echelon form of the augmented matrix.

Example 25. Is the following system consistent? If so, does it have a unique solution?

$$\begin{array}{rrrrrcl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \end{array}$$

Solution. In the course of an earlier example, we obtained the echelon form:

$$\left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Hence, ...

Theorem 26. (Existence and uniqueness theorem) A linear system is **consistent** if and only if an echelon form of the augmented matrix has **no** row of the form

$$\left[\begin{array}{ccc|c} 0 & \ldots & 0 & b \end{array}\right],$$

where b is nonzero.

If a linear system is consistent, then the solution contains either

- a unique solution (when there are no free variables) or
- infinitely many solutions (when there is at least one free variable).

Example 27. For what values of h will the following system be consistent?

$$\begin{array}{rrcl} 3x_1 & -9x_2 & = & 4 \\ -2x_1 & +6x_2 & = & h \end{array}$$

Solution. We perform row reduction to find the echelon form:

$$\left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \rightsquigarrow$$

The system is consistent if and only if h ...

Armin Straub
astraub@illinois.edu

Brief summary of what we learned so far

- Each linear system corresponds to an augmented matrix.
- Using Gaussian elimination (i.e. row reduction to echelon form) on the augmented matrix of a linear system, we can
  - read off, whether the system has no, one, or infinitely many solutions;
  - find all solutions by back-substitution.
- We can continue row reduction to the reduced echelon form.
  - This form is unique!
  - Solutions to the linear system can now be just read off.

Note. Besides for solving linear systems, Gaussian elimination has other important uses, such as computing determinants or inverses of matrices.

A recipe to solve linear systems (Gauss–Jordan elimination)

(1) Write the augmented matrix of the system.

(2) Row reduce to obtain an equivalent augmented matrix in echelon form.
Decide whether the system is consistent. If not, stop; otherwise go to the next step.

(3) Continue row reduction to obtain the reduced echelon form.

(4) Express this final matrix as a system of equations.

(5) Declare the free variables and state the solution in terms of these.

Questions to check our understanding

- On an exam, you are asked to find all solutions to a system of linear equations. You find exactly two solutions. Should you be worried?
- True or false?
  - There is no more than one pivot in any row.
  - There is no more than one pivot in any column.
  - There cannot be more free variables than pivot variables.

Armin Straub
astraub@illinois.edu