Math 415 - Midterm 2

Thursday, October 23, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

Problem 0. ***[1 point]*** Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Babak, Mahmood, Michael, Nathan, Tigran, Travis).

Section:	TA:

To be completed by the grader:

0	1	2	3	4	5	6	Shorts	$\sum$
/1	/?	/?	/?	/?	/?	/?	/?	/?

Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of ? pages. Take a moment to make sure you have all pages.
- You have 75 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

(a) Find a basis for $\mathrm{Nul}(A)$.
(b) Find a basis for $\mathrm{Col}(A^T)$.
(c) Determine the dimension of $\mathrm{Col}(A)$ and the dimension of $\mathrm{Nul}(A^T)$.

Problem 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Find the matrix A which represents T with respect to the following bases:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for } \mathbb{R}^2, \quad \text{and} \quad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \text{ for } \mathbb{R}^3.$$

Problem 3. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

(a) Draw a directed graph with numbered edges and nodes, whose edge-node incidence matrix is A.
(b) Find a basis for $\mathrm{Col}(A^T)$ by choosing a spanning tree of this graph. (This question is not relevant for the second midterm exam!)
(c) Use a property of the graph (briefly explain!) to find a basis for $\mathrm{Nul}(A)$.
(d) Use a property of the graph (briefly explain!) to find a basis for $\mathrm{Nul}(A^T)$.

Problem 4. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - b = c \right\}$.

(a) Write V as a span.
(b) Find a basis for the orthogonal complement of V.

Problem 5. Let $\mathbb{P}_2$ be the vector space of all polynomials of degree up to 2, and let V be the subspace of polynomials $p(t)$ with the property that

$$\int_0^2 p(t)\mathrm{d}t = 0.$$

Find a basis for V. [*Hint*: Write $p(t) = a + bt + ct^2$ and use the integral condition to get a condition on the coefficients of $p(t)$.]

Problem 6. Let $\mathbb{P}_3$ be the vector space of all polynomials of degree up to 3, and let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by

$$T(p(t)) = tp'(t) - 2p(t).$$

(a) Which matrix A represents T with respect to the standard bases?
(b) Find a basis for the null space of A.
[*Optional but recommended*: Interpret in terms of polynomials!]

Problem 7. Let a, b be in $\mathbb{R}$. Consider the three vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} b \\ 1 \\ b \end{bmatrix}.$$

(a) For which values of a and b is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ a basis of $\mathbb{R}^3$?
(b) For which values of a and b does $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ have dimension 2?

Problem 8. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ -1 & -1 & -3 \end{bmatrix}$.

(a) Under which condition(s) on $\boldsymbol{b}$ has the system $A\boldsymbol{x} = \boldsymbol{b}$ a solution?
(b) Find a basis for $\mathrm{Nul}(A)$.
(c) Note that $A \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ -8 \end{bmatrix}$. Find all solutions to $A\boldsymbol{x} = \begin{bmatrix} 0 \\ -8 \\ -8 \end{bmatrix}$.

SHORT ANSWERS

[?? points overall, 3 points each]

Instructions: The following problems have a short answer. No reason needs to be given. If the problem is multiple choice, circle the correct answer (there is always exactly one correct answer).

Short Problem 1. Give a precise definition of what it means for vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ to be linearly independent.

Short Problem 2. Write down a basis for the orthogonal complement of $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Short Problem 3. Let A be an 4×3 matrix, whose row space has dimension 2. What is the dimension of $\operatorname{Nul}(A)$?

Short Problem 4. Let A be an 3×3 matrix, whose column space has dimension 3. If $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$, what can you say about the number of solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$?

Short Problem 5. Let A be an 3×5 matrix of rank 2. Is it possible to find two linearly independent vectors that are orthogonal to the column space of A? For the row space?

(a) Possible for both.
(b) Possible only for the column space.
(c) Possible only for the row space.
(d) Not possible in either case.
(e) Not enough information to decide.

Short Problem 6. Let A be a matrix, and let B be its row reduced echelon form. Which of the following is true for any such matrices?

(a) $\operatorname{Col}(A) = \operatorname{Col}(B)$ and $\operatorname{Col}(A^T) = \operatorname{Col}(B^T)$
(b) $\operatorname{Col}(A) = \operatorname{Col}(B)$ and $\operatorname{Col}(A^T) \neq \operatorname{Col}(B^T)$
(c) $\operatorname{Col}(A) \neq \operatorname{Col}(B)$ and $\operatorname{Col}(A^T) = \operatorname{Col}(B^T)$
(d) $\operatorname{Col}(A) \neq \operatorname{Col}(B)$ and $\operatorname{Col}(A^T) \neq \operatorname{Col}(B^T)$
(e) None of these are true for all such matrices.

Short Problem 7. Let A be a 5×4 matrix. Suppose that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has the solution set

$$\left\{ \begin{bmatrix} 1 - c + d \\ c \\ 3 - 2d \\ d \end{bmatrix} \; : \; c, d \text{ in } \mathbb{R} \right\}.$$

(a) Give a basis for the null space of A.
(b) What is the rank of A?

Short Problem 8. The linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ if and only if ...

(a) $\boldsymbol{b}$ is orthogonal to $\operatorname{Col}(A)$.
(b) $\boldsymbol{b}$ is orthogonal to $\operatorname{Col}(A^T)$.
(c) $\boldsymbol{b}$ is orthogonal to $\operatorname{Nul}(A)$.
(d) $\boldsymbol{b}$ is orthogonal to $\operatorname{Nul}(A^T)$.
(e) Neither of these guarantees a solution.

Short Problem 9. Let A be the edge-node incidence matrix of a directed graph. Suppose that this graph is not a tree (that is, the graph contains at least one loop). What can you say about the rows of A?