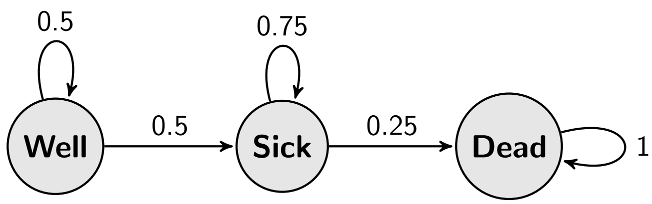
Preparation problems for the discussion sections on December 4th and 9th

1. *Suppose there is an epidemic in which, every month, half of those who are well become sick, and a quarter of those who are sick become dead. Set up a 3×3 transition matrix and find the long term equilibrium (steady state).*

Solution: The transition graph is as follows:



Hence, the transition matrix is

$$A = \begin{bmatrix} .5 & 0 & 0 \\ .5 & .75 & 0 \\ 0 & .25 & 1 \end{bmatrix}.$$

Let

$$u_t = \begin{bmatrix} u_{t,W} & u_{t,S} & u_{t,D} \end{bmatrix},$$

where $u_{t,W}$ is the percentage of people that are well, $u_{t,S}$ is the percentage of people that are sick and $u_{t,D}$ the percentage of people that have died, at time t. Note that

$$u_{t+1} = Au_t.$$

The long term equlibrium is a state u_∞ such that

$$Au_\infty = u_\infty.$$

For that we need to find an eigenvector of A to the eigenvalue 1:

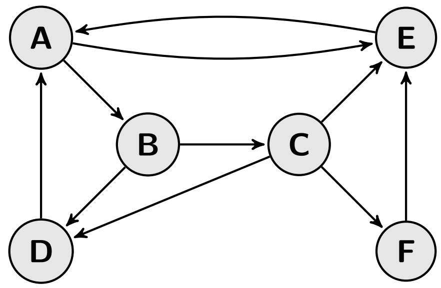
$$\left[\begin{array}{ccc|c} -.5 & 0 & 0 & 0 \\ .5 & -.25 & 0 & 0 \\ 0 & .25 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -.5 & 0 & 0 & 0 \\ 0 & -.25 & 0 & 0 \\ 0 & .25 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -.5 & 0 & 0 & 0 \\ 0 & -.25 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the eigenspace of $\lambda = 1$ is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since this vector is already scaled so that its entries add up to 1, we conclude

$$u_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This means all people will die from the epidemic.

2. *Find the PageRank vector for the following system of webpages, and rank the six webpages accordingly:*



Solution: The PageRank matrix (that is, the transition matrix for the random surfer) is

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}.$$

The PageRank vector is an eigenvector of T to the eigenvalue 1. We row reduce $T-I$ as follows (do it!):

$$\left[\begin{array}{cccccc|c} -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ .5 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & -1 & 0 & 0 & 0 & 0 \\ 0 & .5 & \frac{1}{3} & -1 & 0 & 0 & 0 \\ .5 & 0 & \frac{1}{3} & 0 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 1 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence an eigenvector of T to the eigenvalue 1 is given by

$$\begin{bmatrix} 12 \\ 6 \\ 3 \\ 4 \\ 8 \\ 1 \end{bmatrix},$$

and the corresponding normalized PageRank vector is

$$\frac{1}{34}\begin{bmatrix} 12 \\ 6 \\ 3 \\ 4 \\ 8 \\ 1 \end{bmatrix}.$$

For the purpose of ranking this normalizing is immaterial (and it is OK if you don't normalize PageRank vectors on the exam). The ranking is A, E, B, D, C, F.

3. *Solve the differential equation*

$$\frac{du}{dt} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u$$

with initial condition $u(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solution: Set $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. We first diagonalize A. We start by calculating the eigenvalues of A:

$$\det(A - \lambda I) = \begin{bmatrix} (1-\lambda) & -1 \\ -1 & (1-\lambda) \end{bmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 + 1 = \lambda(\lambda - 2).$$

Hence, the eigenvalues of A are $0, 2$. We now calculate the corresponding eigenvectors. For $\lambda = 0$,

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 0. For $\lambda = 2$,

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A to the eigenvalue 2. Set

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then $A = PDP^{-1}$. Note that

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The solution to the above differential equation with initial condition is

$$\begin{aligned} u(t) &= e^{At}u(0) = Pe^{Dt}P^{-1}u(0) \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2 + e^{2t} \\ 2 - e^{2t} \end{bmatrix}. \end{aligned}$$