Preparation problems for the discussion sections on November 18th and 20th

1. *For each of the following matrices, determine the eigenvalues of the matrix and for each eigenvalue, determine (a basis for) the eigenspace that is associated to that eigenvalue.*

a. $\begin{bmatrix} 4 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$,

b. $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$,

c. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution:

a. For instance, by expanding along the second column, we find that

$$\det \begin{bmatrix} 4-\lambda & 0 & -2 \\ 1 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda)(4-\lambda).$$

Hence, the eigenvalues of A are 2, 4, and 1. For $\lambda = 2$:

$$\begin{bmatrix} 2 & 0 & -2 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2\to R2-1/2R1, R1\to 1/2R1, R2\to -R2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 4$:

$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R2\to R2+R3, R1\to R1-R3, R3\to -1/2R2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 1$:

$$\begin{bmatrix} 3 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2\to R2-2R3, R1\to R1+2R3, R1\to R1-3R2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

b. We have:

$$\det \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix} = (3-\lambda)(-3-\lambda) - 16 = \lambda^2 - 25 = (\lambda-5)(\lambda+5)$$

Hence, the eigenvalues of A are 5 and -5. For $\lambda = 5$:

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \xrightarrow{R2\to R2+2R1, R1\to -1/2R1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$.

For $\lambda = -5$:

$$\begin{bmatrix}8 & 4\\4 & 2\end{bmatrix} \xrightarrow{R2\to R2-1/2R1, R1\to 1/8R1} \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{\begin{bmatrix}-1\\2\end{bmatrix}\right\}$.

c. We have:

$$\det\begin{bmatrix}1-\lambda & 1 & 1\\1 & 1-\lambda & 1\\1 & 1 & 1-\lambda\end{bmatrix} = (1-\lambda)((1-\lambda)(1-\lambda)-1)-(-\lambda)+(1-(1-\lambda))$$
$$= (1-\lambda)(-\lambda)(2-\lambda)+2\lambda = -\lambda((1-\lambda)(2-\lambda)-2) = \lambda^2(3-\lambda)$$

Hence, the eigenvalues of A are 0 and 3. For $\lambda = 0$:

$$\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} \xrightarrow{R2\to R2-R1, R3\to R3-R1} \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$.

For $\lambda = 3$:

$$\begin{bmatrix}-2 & 1 & 1\\1 & -2 & 1\\1 & 1 & -2\end{bmatrix} \xrightarrow{R1\leftrightarrow R3, R2\to R2-R1, R3\to R3+2R1, R3\to R3+R2} \begin{bmatrix}1 & 1 & -2\\0 & -3 & 3\\0 & 0 & 0\end{bmatrix} \xrightarrow{R2\to -1/3R2, R1\to R1-R2} \begin{bmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

2. *Let*

$$A = \begin{bmatrix}2 & 1 & 0 & 0\\0 & 2 & 1 & 0\\0 & 0 & 2 & 1\\0 & 0 & 0 & 3\end{bmatrix}, \quad B = \begin{bmatrix}2 & 0 & 0 & 0\\0 & 2 & 1 & 0\\0 & 0 & 2 & 1\\0 & 0 & 0 & 3\end{bmatrix} \quad C = \begin{bmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 2 & 1\\0 & 0 & 0 & 3\end{bmatrix}$$

Determine the eigenvalues of A, B, C and, for each eigenvalue, determine the eigenspace that is associated to that eigenvalue.

Solution: For A, we have:

$$\det\begin{bmatrix}2-\lambda & 1 & 0 & 0\\0 & 2-\lambda & 1 & 0\\0 & 0 & 2-\lambda & 1\\0 & 0 & 0 & 3-\lambda\end{bmatrix} = (2-\lambda)^3(3-\lambda)$$

Hence, the eigenvalues of A are 2 and 3. For $\lambda = 2$:

$$\begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{R4\to R4-R3} \begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 3$:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For B, we have:

$$\det \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} = (2-\lambda)^3(3-\lambda)$$

Hence, the eigenvalues of B are 2 and 3. For $\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4 \to R4 - R3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 3$:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For C, we have:

$$\det \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} = (2-\lambda)^3(3-\lambda)$$

Hence, the eigenvalues of C are 2 and 3. For $\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4 \to R4 - R3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 3$:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

3. *(This question is not yet relevant for the third midterm exam.) Let*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Solution: We have to find the eigenvalues and corresponding eigenspaces. We have:

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 1 = -\lambda(2-\lambda)$$

Hence, the eigenvalues of A are 0 and 2. For $\lambda = 0$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 2$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{R2 \to R2 + R1, R1 \to -R1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

The columns of P are (linearly independent) eigenvectors of A and D is the diagonal matrix with eigenvalues of A on the main diagonal in the appropriate order (corresponding to columns of P). Therefore:

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$