Preparation problems for the discussion sections on October 14th and 16th

1. *Let* $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Find the length of* $\boldsymbol{v}$. *Find a vector* $\boldsymbol{u}$ *in the direction of* $\boldsymbol{v}$ *that has length* 1 . *Find a vector* $\boldsymbol{w}$ *that is orthogonal to* $\boldsymbol{v}$.

Solution: The length of $\mathbf{v}$ is $\sqrt{1^2+1^2} = \sqrt{2}$. Since $\mathbf{u} = a\mathbf{v}$, we have to find a so that length of $\mathbf{u}$ is 1. So:

$$\sqrt{a^2 + a^2} = 1$$

Thus, $a = \frac{1}{\sqrt{2}}$ and we have:

$$\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For a vector $\mathbf{y}$ orthogonal to $\mathbf{v}$, we need to find $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ such that

$$0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1 + y_2$$

One pair y_1, y_2 that satisfies the equation is $1, -1$. So the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is orthogonal to $\mathbf{v}$.

2. *Let* $\boldsymbol{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, *and* $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. *Find real numbers* c_1, c_2 *such that*

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2.$$

Solution: Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal (i.e. $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$), we have that if

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

for some real number c_1, c_2, then

$$\mathbf{u}_1 \cdot \mathbf{v} = c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_1 \cdot \mathbf{u}_2 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1$$

and

$$\mathbf{u}_2 \cdot \mathbf{v} = c_1\mathbf{u}_2 \cdot \mathbf{u}_1 + c_2\mathbf{u}_2 \cdot \mathbf{u}_2 = c_2\mathbf{u}_2 \cdot \mathbf{u}_2.$$

Hence

$$c_1 = \frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{5}{\sqrt{2}}.$$

and

$$c_2 = \frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{-1}{\sqrt{2}}.$$

(Note: you can also solve the system

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 = \mathbf{v},$$

to find the same answer. But observe that the above approach becomes much simpler if you are working with n orthogonal vectors in $\mathbb{R}^n$.)

3. *Let* $V = \{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a+b+c+d=0\}$ *be a subspace of* $\mathbb{R}^4$.

(a) *Find a basis for* V.
(b) *Find a vector that is orthogonal to* V.
(c) *Can you find two linearly independent vectors that are orthogonal to* V*?*

Solution:

(a) We have:

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = -b-c-d \right\} = \left\{ \begin{bmatrix} -b-c-d \\ b \\ c \\ d \end{bmatrix} : b,c,d \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for V. (If you ignore the first entry, it is easy to see that these vectors are linearly independent)

(Alternatively: note that, by definition, $V = \text{Nul}([1,1,1,1])$. And for any null space, we know how to find a basis.)

(b) Let

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, $V = \text{Col}(A)$ and the orthogonal complement of V is $\text{Nul}(A^T)$. We have:

$$A^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2-R1, R3 \to R3-R1} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2+R3, R1 \to R1+R2} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Thus,

$$\text{Nul}(A^T) = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a = b = c = d \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

In particular, the dimension of orthogonal complement of V is 1 and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to V.

(This was too much work! (But a good illustration that there is many paths to Rome.) Note again that, by definition, $V = \text{Nul}([1,1,1,1])$. Hence, the orthogonal complement is $\text{Col}([1,1,1,1]^T)$, and we immediately find the vector $[1,1,1,1]^T$ as orthogonal to V.)

(Further note that V is actually defined, right away, as those vectors that are orthogonal to $[1,1,1,1]^T$. Make sure that you can see that by writing out the inner product!)

(c) No, as we showed in part (b), the dimension of the orthogonal complement of V is 1 so we cannot find two linearly independent vectors in the orthogonal complement of V.

Note: a vector $\mathbf{v}$ is orthogonal to V if and only of $\mathbf{v}$ is in the orthogonal complement of V.

4. *Let* $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

(a) *Find an echelon form U of A. What are the column spaces $Col(A)$, $Col(U)$? Are they equal?*
(b) *Find a basis for $Col(U)$ and a basis for $Col(A)$.*
(c) *What are the row spaces $Col(A^T)$, and $Col(U^T)$. Are they equal?*
(d) *Find a basis for the row space of A, $Col(A^T)$.*

(a) We have:

$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 2 \\ 1 & 2 & 5 \end{bmatrix} \xrightarrow{R2 \to R2-4R1, R3 \to R3-R1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R3 \to R3+2R2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Col}(A)=\text{Span}\left\{\begin{bmatrix}1\\4\\1\end{bmatrix}, \begin{bmatrix}2\\8\\2\end{bmatrix}, \begin{bmatrix}1\\2\\5\end{bmatrix}\right\}$ and $\text{Col}(U)=\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}\right\}$. They are not equal since the third entry of any vector in $\text{Col}(U)$ is equal to 0 and in particular, the first column of A is not in $\text{Col}(U)$.

This illustrates the fact that the column space is not preserved by row operations!

(b) Since the first column and the third column are pivot columns, a basis for $\text{Col}(A)$ is $\left\{\begin{bmatrix}1\\4\\1\end{bmatrix}, \begin{bmatrix}1\\2\\5\end{bmatrix}\right\}$; and a basis for $\text{Col}(U)$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}\right\}$.

(c) $\text{Col}(A^T)$ and $\text{Col}(U^T)$ are equal since each row of U is a linear combination of rows of A and vice versa. We have:

$$\text{Col}(A^T) = \text{Col}(U^T) = \text{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\-2\end{bmatrix}\right\}$$

The row space, on the other hand, is preserved by row operations!

(d) Non-zero rows of U form a basis for $\text{Col}(A^T)$. Hence, $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\-2\end{bmatrix}\right\}$ is a basis for $\text{Col}(A^T)$.

5. *Let* $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

(a) *Find a basis for $Nul(B)$.*
(b) *Find two linear independent vectors that are orthogonal to $Nul(B)$.*
(c) *Is there a non-zero vector in $\mathbb{R}^2$ orthogonal to $Col(B)$?*

Solution: a) We bring B to reduced echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R2 - R1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Hence, vectors in $Nul(B)$ are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis of $Nul(B)$.

b) The row space of B is orthogonal to $Nul(B)$. Hence it is enough to find a basis of $Row(B)$.

$$B^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R4 \to R4 - R1} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis of $Row(B)$ (in this case, we could have argued right away that the two vectors are independent because it is only two and they are not multiples of each other). Thus $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$ is linearly independent and each one is orthogonal to $Nul(B)$.

c) By part a) $\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$ are the pivot columns of B and hence form a basis of $Col(B)$. Hence $\dim Col(B) = 2$ and so $\mathbb{R}^2 = Col(B)$. Hence a vector $\mathbf{v}$ that is orthogonal to $Col(B)$, is orthogonal to every vector in $\mathbb{R}^2$. In particular, $\mathbf{v}$ is orthogonal to itself. That is $\mathbf{v} \cdot \mathbf{v} = 0$. But then $\mathbf{v} = 0$. Hence, there is no non-zero vector orthogonal to $Col(B)$.

(Another way to see this, is to note that the orthogonal complement has dimension 0, and so only contains the zero vector.)

6. *Let* $\mathcal{B} := \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$ *be a basis of* $\mathbb{R}^3$. *Let* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *be the linear transformation that maps* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *in* $\mathbb{R}^3$ *to* $\begin{bmatrix} z \\ x \\ y \end{bmatrix}$. *Determine the matrix corresponding to* T *with respect to the bases* $\mathcal{B}$ *and* $\mathcal{B}$.

Solution: We have:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} = 0\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} = (-1)\begin{bmatrix}1\\0\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Therefore, the matrix corresponding to T with respect to the bases $\mathcal{B}$ and $\mathcal{B}$ is:

$$A = \begin{bmatrix}0 & -1 & 0\\1 & -1 & 0\\0 & 1 & 1\end{bmatrix}$$

7. *Let $I\colon \mathbb{P}^3 \to \mathbb{P}^4$ be the linear transformation that maps $p(t)$ to*

$$tp(t) + p'(t)$$

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of $\mathbb{P}^3$ and the basis $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ of $\mathbb{P}^4$. Determine the matrix which represents I with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

Solution: We have:

$$\begin{aligned} I(1) &= t.1 - 0 = 0.1 + 1.t + 0.t^2 + 0.t^3 + 0.t^4 \\ I(t) &= t^2 + 1 = 1.1 + 0.t + 1.t^2 + 0.t^3 + 0.t^4 \\ I(t^2) &= t^3 + 2.t = 0.1 + 2.t + 0.t^2 + 1.t^3 + 0.t^4 \\ I(t^3) &= t^4 + 3.t^2 = 0.1 + 0.t + 3.t^2 + 0.t^3 + 1.t^4 \end{aligned}$$

Therefore, the matrix A that represent I with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ is: (we put coefficients of $I(1), I(t), I(t^2)$, and $I(t^3)$ respectively in the first, second, third, and forth column)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. *True or False? Justify your answers.*

(a) *The map $T\colon \mathbb{R}^2 \to \mathbb{R}$ given by $T\begin{bmatrix} a \\ b \end{bmatrix} = \sqrt{a^2 + b^2}$ is a linear transformation.*

(b) *The map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$ is a linear transformation.*

(c) *If $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^2$ are such that $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ ($\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal) then $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular (geometrically) to each other.*

(d) *Let V be a subspace and $\boldsymbol{u}, \boldsymbol{v}$ be two vectors in V, then $\boldsymbol{v} - \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u}$ is orthogonal to $\boldsymbol{u}$.*

(e) *Let $T : V \to W$ be a linear transformation and $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ be vectors in V. If $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_n)$ are linearly independent then $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are also linearly independent.*

(f) *Let $T : V \to W$ be a linear transformation and $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ be vectors in V. If $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are linearly independent then $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_n)$ are also linearly independent.*

(g) *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. The dimension of the image of T is equal to 2.*

Solution:

(a) False, since we have:

$$T\begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq -T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) True. Since we have:

$$T\begin{bmatrix} a \\ b \end{bmatrix} + T\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} + \begin{bmatrix} -b' \\ a' \end{bmatrix} = \begin{bmatrix} -(b+b') \\ a+a' \end{bmatrix} = T\begin{bmatrix} a+a' \\ b+b' \end{bmatrix}, \quad T\begin{bmatrix} ra \\ rb \end{bmatrix} = \begin{bmatrix} -rb \\ ra \end{bmatrix} = r\begin{bmatrix} -b \\ a \end{bmatrix} = rT\begin{bmatrix} a \\ b \end{bmatrix}$$

(Alternatively, can you see which matrix gives rise, by matrix multiplication, to the same map?)

(c) True, let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^2$. Then:

$$[length(\mathbf{a}-\mathbf{b})]^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = [length(\mathbf{a})]^2+[length(\mathbf{b})]^2$$

Thus by the Pythagorean theorem, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other.

(d) True, since we have:

$$\mathbf{u}\cdot(\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}) = \mathbf{u}\cdot\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\mathbf{u} = 0$$

(e) True, since if $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_n\mathbf{v}_n = 0$ then $x_1T(\mathbf{v}_1) + x_2T(\mathbf{v}_2) + ... + x_nT(\mathbf{v}_n) = 0$, but $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)$ are linearly independent so all x_is are equal to 0.

(f) False, consider $T : \mathbb{R}^3 \to \mathbb{R}$ so that $T(\mathbf{v}) = 0$ (every vector is sent to zero).

(g) False, by (e) we know that the dimension of the image of T is at most 2, but it is not necessarily equal to 2. Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ so that $T(\mathbf{v}) = 0$ (again, every vector is sent to zero).