Review for the final exam

- Bring a **number 2 pencil** to the exam!
- Room assignments for Friday, Dec 12, 7-10pm:
  - if your last name starts with A-J: 114 DKH
  - if your last name starts with K-Z: Foellinger Auditorium

What we have learned

- Basic notions of linear algebra:

  Make sure that you can say, precisely, what each notion means.

  - **vector spaces** and **subspaces**
  - **linear independence**
  - **basis** and **dimension**
  - **linear transformations** (and matrices representing them)
  - **orthogonal** vectors, spaces, matrices, and projections
  - the four **fundamental subspaces** associated with a matrix
- Technical skills:
  - matrix multiplication
  - Gaussian elimination and solving linear systems
  - LU decomposition
  - computing matrix inverses
  - Gram–Schmidt and QR decomposition
  - finding least squares solutions
  - determine (orthonormal) bases of spaces and their orthogonal complements
  - computing determinants
  - eigenvalues, eigenvectors and diagonalization
- Applications
  - finite differences (not on the exam)
  - directed graphs (both null spaces of edge-node incidence matrix)
  - Fourier series
  - linear regression (least squares lines)
  - difference equations (steady state, PageRank)
  - systems of differential equations

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Practice problems

Example 1. Are the following vector spaces?

(a) The set of all functions $f:\mathbb{R}\to\mathbb{R}$ with $f''(0)=7$.

No. Missing the zero function.

(b) The set of all orthogonal 2×2 matrices.

No. Missing the zero matrix.

(c) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1+x_2=0$.

Yes. This is $\mathrm{Nul}(\begin{bmatrix} 1 & 1 \end{bmatrix})$.

(d) The set of all eigenvectors of a matrix A.

No. Take, for instance, $A=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(e) The set of solutions $y(x)$ of $y''+7y'-y=0$.

Yes. For instance, if $y_1''+7y_1'-y_1=0$ and $y_2''+7y_2'-y_2=0$, then $(y_1+y_2)''+7(y_1+y_2)'-(y_1+y_2)=0$.

Example 2. Decide if each criterion is true or false.

An $n\times n$ matrix A is invertible if and only if...

(a) the columns of A are independent.

True.

(b) 0 is not an eigenvalue of A.

True.

(c) A has no zero column.

False. Take, for instance, $A=\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

(d) A has no free variables.

True.

(e) A is row equivalent to I.

True.

(f) A has n independent eigenvectors.

False. Take, for instance, $A=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 3. Decide if each criterion is true or false.

The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if...

(a) an echelon form of $[\ A\ \ \boldsymbol{b}\ ]$ has no row of the form $[\ 0\ \ \cdots\ \ 0\ \ \beta\ ]$ with $\beta \neq 0$.

True.

(b) $\boldsymbol{b}$ is in $\operatorname{Col}(A^T)$.

False. ($\boldsymbol{b}$ is in $\operatorname{Col}(A)$ would be true.)

(c) A has no free variables.

False. Take, for instance, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(d) $\boldsymbol{b}$ is orthogonal to $\operatorname{Nul}(A^T)$.

True.

Example 4. What is $\begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 8 \\ 0 & 3 & 3 & 1 \\ 2 & -7 & 0 & 5 \end{vmatrix}$?

Solution. The determinant is 0 because the matrix is not invertible (second row is a multiple of the first).

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Example 5. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

- Diagonalize $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$:
  - $\begin{vmatrix} -\lambda & -2 \\ -4 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8$, so the eigenvalues are $-2, 4$
  - $\lambda = 4$ has eigenspace $\operatorname{Nul}\left(\begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$
  - $\lambda = -2$ has eigenspace $\operatorname{Nul}\left(\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned}
\boldsymbol{y} &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\
&= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}
\end{aligned}$$

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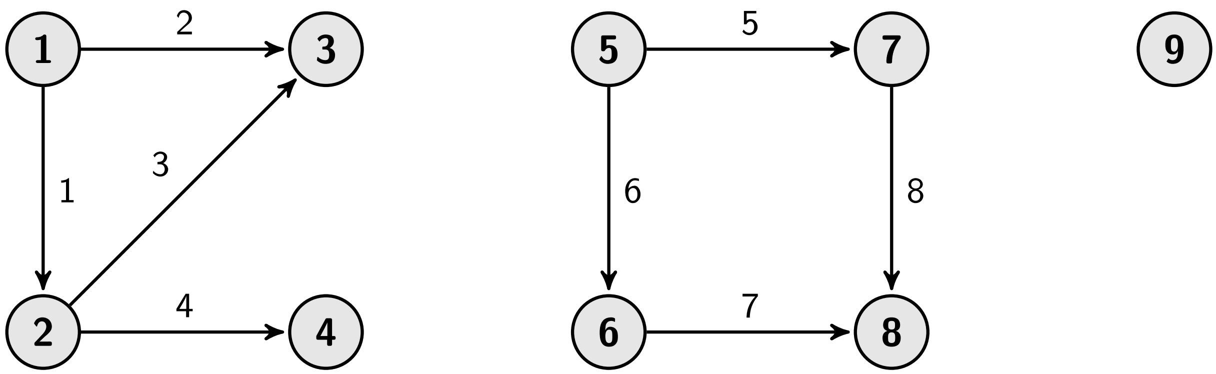
Example 6. Determine a basis for $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^T)$, where A is the edge-node incidence matrix of the directed graph below.



Solution.

- Basis for $\mathrm{Nul}(A)$ from connected subgraphs.

  For each connected subgraph, get a basis vector $\boldsymbol{x}$ that assigns 1 to all nodes in that subgraph, and 0 to all other nodes.

  Basis for $\mathrm{Nul}(A)$: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

- Basis for $\mathrm{Nul}(A^T)$ from (independent) loops.

  For each (independent) loop, get a basis vector $\boldsymbol{y}$ that assigns 1 and -1 (depending on direction) to the edges in that loop, and 0 to all other edges.

  Basis for $\mathrm{Nul}(A^T)$: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

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Example 7. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$.

(a) Determine the LU decomposition of A.

(b) What is $\det(A)$?

Solution.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R2 \to R2 - R1}{R3 \to R3 + 2R1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & 3 \end{bmatrix}$ $\qquad L = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ -2 & * & 1 \end{bmatrix}$

$\underset{\rightsquigarrow}{R3 \to R3 - 3R1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 6 \end{bmatrix} = U$ $\qquad L = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ -2 & 3 & 1 \end{bmatrix}$

The LU decomposition is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 6 \end{bmatrix}$.

In the exam: check!!!

(b) $\det(A) = \det(L)\det(U) = 1 \cdot (1 \cdot 1 \cdot 6) = 6$

Example 8. Suppose that, with respect to the bases

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ of } \mathbb{R}^2, \text{ and } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ of } \mathbb{R}^3,$$

the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ is represented by $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix}$.

(a) What exactly does the matrix encode?

(b) What is $T\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right)$?

Solution.

(a) $T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $T\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

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