Welcome back!

- The final exam is on Friday, December 12, 7-10pm

  If you have a conflict (overlapping exam, or more than 2 exams within 24h), please email Mahmood until Sunday to sign-up for the Monday conflict.

- What are the eigenspaces of $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$?
  - $\lambda = 1$ has eigenspace $\mathrm{Nul}\left(\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$
  - $\lambda = 3$ has eigenspace $\mathrm{Nul}\left(\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$
  - INCORRECT: eigenspace $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Transition matrices

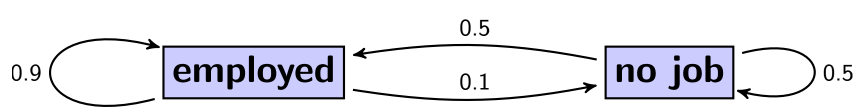
Powers of matrices can describe transition of a system.

Example 1. (review)

- Fibonacci numbers F_n: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$
- $F_{n+1} = F_n + F_{n-1} \implies \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$
- Hence: $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$

Example 2. Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed loose their job.
What is the unemployment rate in the long term equilibrium?

Solution.



x_t: proportion of population employed at time t (in years)

y_t: proportion of population unemployed at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.9x_t + 0.5y_t \\ 0.1x_t + 0.5y_t \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$ is a **Markov matrix**. Its columns add to 1 and it has no negative entries.

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$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$.

In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

Eigenspace of $\lambda = 1$: $\operatorname{Nul}\left(\begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix} \right) = \operatorname{span}\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$.

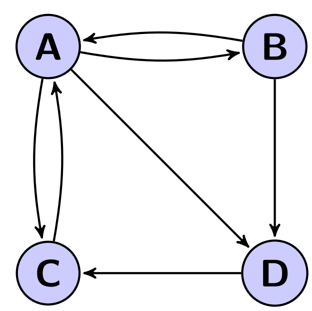
Hence, the unemployment rate in the long term equilibrium is $1/6$.

Page rank

Google's success is based on an algorithm to rank webpages, the **Page rank**, named after Google founder Larry Page.

The basic idea is to determine how likely it is that a web user randomly gets to a given webpage. The webpages are then ranked by these probabilities.

Example 3. Suppose the internet consisted of only the four webpages A, B, C, D linked as in the following graph:



Imagine a surfer following these links at random.

For the probability $\mathrm{PR}_n(A)$ that she is at A (after n steps), we add:

- the probability that she was at B (at exactly one time step before), and left for A, (that's $\mathrm{PR}_{n-1}(B) \cdot \frac{1}{2}$)
- the probability that she was at C, and left for A,
- the probability that she was at D, and left for A.

- Hence: $\mathrm{PR}_n(A) = \mathrm{PR}_{n-1}(B) \cdot \frac{1}{2} + \mathrm{PR}_{n-1}(C) \cdot \frac{1}{1} + \mathrm{PR}_{n-1}(D) \cdot \frac{0}{1}$
- $\begin{bmatrix} \mathrm{PR}_n(A) \\ \mathrm{PR}_n(B) \\ \mathrm{PR}_n(C) \\ \mathrm{PR}_n(D) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{PR}_{n-1}(A) \\ \mathrm{PR}_{n-1}(B) \\ \mathrm{PR}_{n-1}(C) \\ \mathrm{PR}_{n-1}(D) \end{bmatrix}$
- The **PageRank vector** $\begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix} = \begin{bmatrix} \mathrm{PR}_\infty(A) \\ \mathrm{PR}_\infty(B) \\ \mathrm{PR}_\infty(C) \\ \mathrm{PR}_\infty(D) \end{bmatrix}$ is the long-term equilibrium.

  It is an eigenvector of the Markov matrix with eigenvalue 1.
- $\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

  $\Longrightarrow$ eigenspace of $\lambda = 1$ spanned by $\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$

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$$\implies \begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix} = \frac{3}{16}\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$$ This the PageRank vector.

- The corresponding ranking of the webpages is A, C, D, B.

Practice problems

Problem 1. Can you see why 1 is an eigenvalue for every Markov matrix?

Problem 2. (just for fun) The real web contains pages which have no outgoing links. In that case, our random surfer would get "stuck" (the transition matrix is not a Markov matrix). Do you have an idea how to deal with this issue?

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