Review

- $\boldsymbol{v}\cdot\boldsymbol{w}=\boldsymbol{v}^T\boldsymbol{w}=v_1w_1+\ldots+v_nw_n$, the **inner product** of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$
  - **Length** of $\boldsymbol{v}$: $\|\boldsymbol{v}\|=\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}=\sqrt{v_1^2+\ldots+v_n^2}$
  - **Distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$: $\|\boldsymbol{v}-\boldsymbol{w}\|$
- $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if $\boldsymbol{v}\cdot\boldsymbol{w}=0$.
  - This simple criterion is equivalent to Pythagoras theorem.

Example 1. The vectors $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$

- are orthogonal to each other, and
- have length 1.

We are going to call such a basis **orthonormal** soon.

Theorem 2. Suppose that $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n$ are nonzero and (pairwise) orthogonal. Then $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n$ are independent.

Proof. Suppose that

$$c_1\boldsymbol{v}_1+\ldots+c_n\boldsymbol{v}_n=\boldsymbol{0}.$$

Take the dot product of $\boldsymbol{v}_1$ with both sides:

$$\begin{aligned}0&=\boldsymbol{v}_1\cdot(c_1\boldsymbol{v}_1+\ldots+c_n\boldsymbol{v}_n)\\&=c_1\boldsymbol{v}_1\cdot\boldsymbol{v}_1+c_2\boldsymbol{v}_1\cdot\boldsymbol{v}_2+\ldots+c_n\boldsymbol{v}_1\cdot\boldsymbol{v}_n\\&=c_1\boldsymbol{v}_1\cdot\boldsymbol{v}_1=c_1\|\boldsymbol{v}_1\|^2\end{aligned}$$

But $\|\boldsymbol{v}_1\|\neq0$ and hence $c_1=0$.

Likewise, we find $c_2=0$, …, $c_n=0$. Hence, the vectors are independent. □

Example 3. Let us consider $A=\begin{bmatrix}1&2\\2&4\\3&6\end{bmatrix}$.

Find $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$. Observe!

Solution.

$\mathrm{Nul}(A)=\mathrm{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$ — can you see it? if not, do it!

$\mathrm{Col}(A^T)=\mathrm{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

The two basis vectors are orthogonal! $\begin{bmatrix}-2\\1\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}=0$

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Example 4. Repeat for $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -3 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\mathrm{Col}(A^T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ The 2 vectors form a basis.

Again, the vectors are orthogonal!

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$$

Note: Because $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to both basis vectors, it is orthogonal to every vector in the row space.

Vectors in $\mathrm{Nul}(A)$ are orthogonal to vectors in $\mathrm{Col}(A^T)$.

The fundamental theorem, second act

Definition 5. Let W be a subspace of $\mathbb{R}^n$, and $\boldsymbol{v}$ in $\mathbb{R}^n$.

- $\boldsymbol{v}$ is **orthogonal** to W, if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ for all $\boldsymbol{w}$ in W.

  ($\iff$ $\boldsymbol{v}$ is orthogonal to each vector in a basis of W)

- Another subspace V is **orthogonal** to W, if every vector in V is orthogonal to W.
- The **orthogonal complement** of W is the space $W^\perp$ of all vectors that are orthogonal to W.

  Exercise: show that the orthogonal complement is indeed a vector space.

Example 6. In the previous example, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

We found that

$$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathrm{Col}(A^T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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are orthogonal subspaces.

Indeed, $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are orthogonal complements.

Why? Because $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal, hence independent, and hence a basis of all of $\mathbb{R}^3$.

Remark 7. Recall that, for an $m \times n$ matrix A, $\mathrm{Nul}(A)$ lives in $\mathbb{R}^n$ and $\mathrm{Col}(A)$ lives in $\mathbb{R}^m$. Hence, they cannot be related in a similar way.

In the previous example, they happen to be both subspaces of $\mathbb{R}^3$:

$$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

But these spaces are not orthogonal: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq 0$

Theorem 8. (Fundamental Theorem of Linear Algebra, Part I)

Let A be an $m \times n$ matrix of rank r.

- $\dim \mathrm{Col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \mathrm{Col}(A^T) = r$ (subspace of $\mathbb{R}^n$)
- $\dim \mathrm{Nul}(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim \mathrm{Nul}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

Theorem 9. (Fundamental Theorem of Linear Algebra, Part II)

- $\mathrm{Nul}(A)$ is orthogonal to $\mathrm{Col}(A^T)$. (both subspaces of $\mathbb{R}^n$)

  Note that $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$.

  Hence, the two spaces are orthogonal complements in $\mathbb{R}^n$.

- $\mathrm{Nul}(A^T)$ is orthogonal to $\mathrm{Col}(A)$.

  Again, the two spaces are orthogonal complements.

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