Review

- A **linear map** $T: V \to W$ satisfies $T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y})$.
- $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$ is linear. (A an $m \times n$ matrix)
  - A is the matrix representing T w.r.t. the standard bases

    For instance: $T(\boldsymbol{e}_1) = A\boldsymbol{e}_1 = 1^{\text{st}}$ column of A $\qquad \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
- Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for V, and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ a basis for W.
  - The matrix representing T w.r.t. these bases encodes in column j the coefficients of $T(\boldsymbol{x}_j)$ expressed as a linear combination of $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$.
  - For instance: let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be reflection through the x-y-plane, that is, $(x, y, z) \mapsto (x, y, -z)$.

    The matrix representing T w.r.t. the basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$.

    $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

    $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Example 1. Let $T: \mathbb{P}_3 \to \mathbb{P}_2$ be the linear map given by

$$T(p(t)) = \frac{\mathrm{d}}{\mathrm{d}t} p(t).$$

What is the matrix A representing T with respect to the standard bases?

Solution. The bases are

$$1,\ t,\ t^2,\ t^3 \text{ for } \mathbb{P}_3, \qquad 1,\ t,\ t^2 \text{ for } \mathbb{P}_2.$$

The matrix A has 4 columns and 3 rows.

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The first column encodes $T(1)=0$ and hence is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

For the second column, $T(t)=1$ and hence it is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For the third column, $T(t^2)=2t$ and hence it is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

For the last column, $T(t^3)=3t^2$ and hence it is $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

In conclusion, the matrix representing T is

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Note: By the way, what is the null space of A?

The null space has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The corresponding polynomial is $p(t)=1$.

No surprise here: differentation kills precisely the constant polynomials.

Note: Let us differentiate $7t^3-t+3$ using the matrix A.

- First: $7t^3-t+3$ w.r.t. standard basis: $\begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix}$.
- $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 21 \end{bmatrix}$
- $\begin{bmatrix} -1 \\ 0 \\ 21 \end{bmatrix}$ in the standard basis is $-1+21t^2$.

Orthogonality

The inner product and distances

Definition 2. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \;=\; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 3. For instance,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = 1 - 2 - 6 = -7.$$

Definition 4.

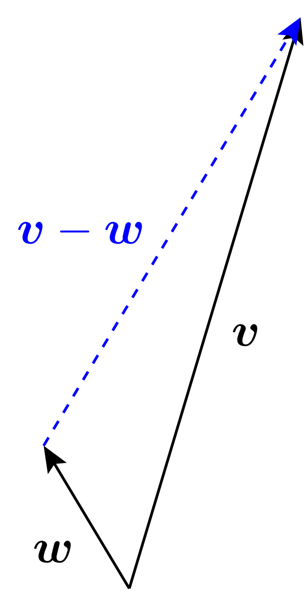
- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \;=\; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

This is the distance to the origin.

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\mathrm{dist}(\boldsymbol{v}, \boldsymbol{w}) \;=\; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 5. For instance, in $\mathbb{R}^2$,

$$\mathrm{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) \;=\; \left\| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

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Orthogonal vectors

Definition 6. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$

How is this related to our understanding of right angles?

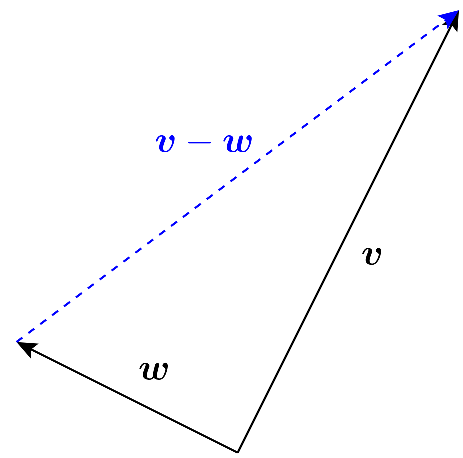
Pythagoras:

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$\iff \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2$

$\iff \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \underbrace{(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w})}_{\boldsymbol{v} \cdot \boldsymbol{v} - 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}}$

$\iff \boldsymbol{v} \cdot \boldsymbol{w} = 0$



Example 7. Are the following vectors orthogonal?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1 \cdot (-2) + 2 \cdot 1 = 0$. So, yes, they are orthogonal.

(b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot (-2) + 2 \cdot 1 + 1 \cdot 1 = 1$. So not orthogonal.

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