Review

- $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is a **basis** of V if the vectors span V and are independent.
- To obtain a basis for $\operatorname{Nul}(A)$, solve $A\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 \\ 6 & 12 & 15 & 0 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} -2x_2 - 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Hence, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ form a basis for $\operatorname{Nul}(A)$.

- To obtain a basis for $\operatorname{Col}(A)$, take the pivot columns of A.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ form a basis for $\operatorname{Col}(A)$.

- Row operations do not preserve the column space.

For instance, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{R1 \leftrightarrow R2}{\rightsquigarrow} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- On the other hand: row operations do preserve the null space.

Why? Recall why/that we can operate on rows to solve systems like $A\boldsymbol{x} = 0$!

Dimension of $\operatorname{Col}(A)$ and $\operatorname{Nul}(A)$

Definition 1. The **rank** of a matrix A is the number of its pivots.

Theorem 2. Let A be an $m \times n$ matrix of rank r. Then:

- $\dim \operatorname{Col}(A) = r$

Why? A basis for $\operatorname{Col}(A)$ is given by the pivot columns of A.

- $\dim \operatorname{Nul}(A) = n - r$ is the number of free variables of A

Why? In our recipe for a basis for $\operatorname{Nul}(A)$, each free variable corresponds to an element in the basis.

- $\dim \operatorname{Col}(A) + \dim \operatorname{Nul}(A) = n$

Why? Each of the n columns either contains a pivot or corresponds to a free variable.

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The four fundamental subspaces

Row space and left null space

Definition 3.

- The **row space** of A is the column space of A^T.

  $\mathrm{Col}(A^T)$ is spanned by the columns of A^T and these are the rows of A.

- The **left null space** of A is the null space of A^T.

  Why "left"? A vector $\boldsymbol{x}$ is in $\mathrm{Nul}(A^T)$ if and only if $A^T\boldsymbol{x} = \mathbf{0}$.

  Note that $A^T\boldsymbol{x} = \mathbf{0} \Longleftrightarrow (A^T\boldsymbol{x})^T = \boldsymbol{x}^T A = \mathbf{0}^T$.

  Hence, $\boldsymbol{x}$ is in $\mathrm{Nul}(A^T)$ if and only if $\boldsymbol{x}^T A = \mathbf{0}$.

Example 4. Find a basis for $\mathrm{Col}(A)$ and $\mathrm{Col}(A^T)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution. We know what to do for $\mathrm{Col}(A)$ from an echelon form of A, and we could likewise handle $\mathrm{Col}(A^T)$ from an echelon form of A^T.

But wait!

Instead of doing twice the work, we only need an echelon form of A:

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Hence, the rank of A is 2.

A basis for $\mathrm{Col}(A)$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$.

Recall that $\mathrm{Col}(A) \neq \mathrm{Col}(B)$. That's because we performed row operations.

However, the row spaces are the same! $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$

The row space is preserved by elementary row operations.

In particular: a basis for $\mathrm{Col}(A^T)$ is given by $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -5 \end{bmatrix}$.

Theorem 5. **(Fundamental Theorem of Linear Algebra, Part I)**

Let A be an $m \times n$ matrix of rank r.

- $\dim \mathrm{Col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \mathrm{Col}(A^T) = r$ (subspace of $\mathbb{R}^n$)
- $\dim \mathrm{Nul}(A) = n - r$ (subspace of $\mathbb{R}^n$) (# of free variables of A)
- $\dim \mathrm{Nul}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

In particular:

The column and row space always have the same dimension!

In other words, A and A^T have the same rank. [i.e. same number of pivots]

Easy to see for a matrix in echelon form

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

but not obvious for a random matrix.

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Linear transformations

Throughout, V and W are vector spaces.

Definition 6. A map $T: V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

In other words, a linear transformation respects addition and scaling:

- $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
- $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$

It also sends the zero vector in V to the zero vector in W:

- $T(\mathbf{0}) = \mathbf{0}$ [because $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$]

Example 7. Let A be an $m \times n$ matrix.

Then the map $T(\boldsymbol{x}) = A\boldsymbol{x}$ is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$.

Why?

Because matrix multiplication is linear:

$A(c\boldsymbol{x} + d\boldsymbol{y}) = cA\boldsymbol{x} + dA\boldsymbol{y}$

The LHS is $T(c\boldsymbol{x} + d\boldsymbol{y})$ and the RHS is $cT(\boldsymbol{x}) + dT(\boldsymbol{y})$.

Example 8. Let $\mathbb{P}_n$ be the vector space of all polynomials of degree at most n. Consider the map $T: \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by

$$T(p(t)) = \frac{\mathrm{d}}{\mathrm{d}t} p(t).$$

This map is linear! Why?

Because differentiation is linear:

$\frac{\mathrm{d}}{\mathrm{d}t}[ap(t) + bq(t)] = a\frac{\mathrm{d}}{\mathrm{d}t}p(t) + b\frac{\mathrm{d}}{\mathrm{d}t}q(t)$

The LHS is $T(ap(t) + bq(t))$ and the RHS is $aT(p(t)) + bT(q(t))$.

Representing linear maps by matrices

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for V.

A linear map $T: V \to W$ is determined by the values $T(\boldsymbol{x}_1), \ldots, T(\boldsymbol{x}_n)$.

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Why?

Take any $\boldsymbol{v}$ in V.

It can be written as $\boldsymbol{v} = c_1\boldsymbol{x}_1 + \ldots + c_n\boldsymbol{x}_n$ because $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is a basis and hence spans V.

Hence, by the linearity of T,

$$T(\boldsymbol{v}) = T(c_1\boldsymbol{x}_1 + \ldots + c_n\boldsymbol{x}) = \quad c_1T(\boldsymbol{x}_1) + \ldots + c_nT(\boldsymbol{x}_n).$$

Definition 9. (From linear maps to matrices)

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for V, and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ a basis for W.

The **matrix representing** T with respect to these bases

- has n columns (one for each of the $\boldsymbol{x}_j$),
- the j-th column has m entries $a_{1,j}, \ldots, a_{m,j}$ determined by

$$T(\boldsymbol{x}_j) = a_{1,j}\boldsymbol{y}_1 + \ldots + a_{m,j}\boldsymbol{y}_m.$$

Example 10. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}.$$

What is the matrix A representing T with respect to the standard bases?

Solution. The standard bases are

$$\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\boldsymbol{x}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{x}_2} \text{ for } \mathbb{R}^2, \qquad \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_3} \text{ for } \mathbb{R}^3.$$

$$\begin{aligned} T(\boldsymbol{x}_1) &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= 1\boldsymbol{y}_1 + 2\boldsymbol{y}_2 + 3\boldsymbol{y}_3 \\ &\Longrightarrow \quad A = \begin{bmatrix} 1 & * \\ 2 & * \\ 3 & * \end{bmatrix} \\ T(\boldsymbol{x}_2) &= \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = 4\boldsymbol{y}_1 + 0\boldsymbol{y}_2 + 7\boldsymbol{y}_3 \\ &\Longrightarrow \quad A = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 7 \end{bmatrix} \end{aligned}$$

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(We did not have time yet to discuss the next example in class, but it will be helpful if your discussion section already meets Tuesdays.)

Example 11. As in the previous example, let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the (same) linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}.$$

What is the matrix B representing T with respect to the following bases?

$$\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{x}_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\boldsymbol{x}_2} \text{ for } \mathbb{R}^2, \qquad \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_3} \text{ for } \mathbb{R}^3.$$

Solution. This time:

$$\begin{aligned}
T(\boldsymbol{x}_1) &= T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 10 \end{bmatrix} \\
&= 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\Longrightarrow B &= \begin{bmatrix} 5 & * \\ -3 & * \\ 5 & * \end{bmatrix} \\
T(\boldsymbol{x}_2) &= T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = -T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 11 \end{bmatrix} \\
&= 7\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 9\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\Longrightarrow B &= \begin{bmatrix} 5 & 7 \\ -3 & -9 \\ 5 & 4 \end{bmatrix}
\end{aligned}$$

can you see it? otherwise: do it!

Tedious, even in this simple example! (But we can certainly do it.)

A matrix representing T encodes in column j the coefficients of $T(\boldsymbol{x}_j)$ expressed as a linear combination of $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$.

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Practice problems

Example 12. Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 7 & 8 & 1 \end{bmatrix}$. Find the dimensions and a basis for all four fundamental subspaces of A.

Example 13. Suppose A is a 5×5 matrix, and that $\boldsymbol{v}$ is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of A.

What can you say about the number of solutions to $A\boldsymbol{x} = \mathbf{0}$?

Solution. Stop reading, unless you have thought about the problem!

Existence of such a $\boldsymbol{v}$ means that the 5 columns of A do not span $\mathbb{R}^5$.

Hence, the columns are not independent.

In other words, A has at most 4 pivots.

So, at least one free variable.

Which means that $A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions.

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