Review

- $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is a **basis** of V if the vectors
  - span V, and
  - are independent.
- The **dimension** of V is the number of elements in a basis.
- The columns of A are linearly independent

  $\iff$ each column of A contains a pivot.

Warmup

Example 1. Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \text{ real} \right\}.$$

Solution.

First, note that

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Is $\dim W = 4$? No, because the third vector is the sum of the first two.

Suppose we did not notice...

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Not a pivot in every column, hence the 4 vectors are dependent.

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[Not necessary here, but:

To get a relation, solve $A\boldsymbol{x} = \mathbf{0}$. Set free variable $x_3 = 1$.

Then $x_4 = 0$, $x_2 = -x_3 = -1$ and $x_1 = -x_2 - 2x_3 = -1$. The relation is

$$-\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}.$$

Precisely, what we "noticed" to begin with.]

Hence, a basis for W is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\dim W = 3$.

It follows from the echelon form that these vectors are independent.

Every set of linearly independent vectors can be extended to a basis.

In other words, let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ be linearly independent vectors in V. If V has dimension d, then we can find vectors $\boldsymbol{v}_{p+1}, \ldots, \boldsymbol{v}_d$ such that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ is a basis of V.

Example 2. Consider

$$H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- Give a basis for H. What is the dimension of H?
- Extend the basis of H to a basis of $\mathbb{R}^3$.

Solution.

- The vectors are independent. By definition, they span H.

  Therefore, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for H.

  In particular, $\dim H = 2$.

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^3$. Why?

  Because a basis for $\mathbb{R}^3$ needs to contain 3 vectors.

  Or, because, for instance, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in H.

  So: just add this (or any other) missing vector!

  By construction, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is independent.

  Hence, this automatically is a basis of $\mathbb{R}^3$.

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Bases for column and null spaces

Bases for null spaces

To find a basis for $\mathrm{Nul}(A)$:

- find the parametric form of the solutions to $A\boldsymbol{x}=\mathbf{0}$,
- express solutions $\boldsymbol{x}$ as a linear combination of vectors with the free variables as coefficients;
- these vectors form a basis of $\mathrm{Nul}(A)$.

Example 3. Find a basis for $\mathrm{Nul}(A)$ with

$$A=\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 0 & 0 & 3 & -6 & -15 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 5 & 13 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix}$$

The solutions to $A\boldsymbol{x}=\mathbf{0}$ are:

$$\boldsymbol{x}=\begin{bmatrix} -2x_2-5x_4-13x_5 \\ x_2 \\ 2x_4+5x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\mathrm{Nul}(A)=\mathrm{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}\right\}$.

These vectors are clearly independent.

If you don't see it, do compute an echelon form! (permute first and third row to the bottom)

Better yet: note that the first vector corresponds to the solution with $x_2=1$ and the other free variables $x_4=0$, $x_5=0$. The second vector corresponds to the solution with $x_4=1$ and the other free variables $x_2=0$, $x_5=0$. The third vector …

Hence, $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for $\mathrm{Nul}(A)$.

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Bases for column spaces

Recall that the columns of A are independent

$\iff$ $A\boldsymbol{x}=\boldsymbol{0}$ has only the trivial solution (namely, $\boldsymbol{x}=0$),

$\iff$ A has no free variables.

A basis for $\mathrm{Col}(A)$ is given by the pivot columns of A.

Example 4. Find a basis for $\mathrm{Col}(A)$ with

$$A=\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are the first and third.

Hence, a basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Warning: For the basis of $\mathrm{Col}(A)$, you have to take the columns of A, not the columns of an echelon form.

Row operations do not preserve the column space.

[For instance, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{R1 \leftrightarrow R2}{\rightsquigarrow} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ have different column spaces (of the same dimension).]

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