Review

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are **linearly dependent** if

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_p\boldsymbol{v}_p = \boldsymbol{0},$$

  and not all the coefficients are zero.
- The columns of A are linearly independent
  $\Longleftrightarrow$ each column of A contains a pivot.
- Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ independent?

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

  So: no, they are dependent! (Coeff's $x_3 = 1$, $x_2 = -2$, $x_1 = 3$)
- Any set of 11 vectors in $\mathbb{R}^{10}$ is linearly dependent.

A basis of a vector space

Definition 1. A set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in V is a **basis** of V if

- $V = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$, and
- the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are linearly independent.

In other words, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in V is a basis of V if and only if every vector $\boldsymbol{w}$ in V can be uniquely expressed as $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_p\boldsymbol{v}_p$.

Example 2. Let $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis of $\mathbb{R}^3$.

It is called the **standard basis**.

Solution.

- Clearly, $\operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} = \mathbb{R}^3$.
- $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are independent, because

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  has a pivot in each column.

Definition 3. V is said to have **dimension** p if it has a basis consisting of p vectors.

Armin Straub
astraub@illinois.edu

This definition makes sense because if V has a basis of p vectors, then every basis of V has p vectors.

Why? (Think of $V = \mathbb{R}^3$.)

A basis of $\mathbb{R}^3$ cannot have more than 3 vectors, because any set of 4 or more vectors in $\mathbb{R}^3$ is linearly dependent.

A basis of $\mathbb{R}^3$ cannot have less than 3 vectors, because 2 vectors span at most a plane (challenge: can you think of an argument that is more "rigorous"?).

Example 4. $\mathbb{R}^3$ has dimension 3.

Indeed, the standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has three elements.

Likewise, $\mathbb{R}^n$ has dimension n.

Example 5. Not all vector spaces have a finite basis. For instance, the vector space of all polynomials has *infinite dimension*.

Its standard basis is $1, t, t^2, t^3, \ldots$

This is indeed a basis, because any polynomial can be written as a unique linear combination: $p(t) = a_0 + a_1 t + \ldots + a_n t^n$ for some n.

Recall that vectors in V form a **basis** of V if they span V and if they are linearly independent. If we know the dimension of V, we only need to check one of these two conditions:

Theorem 6. Suppose that V has dimension d.

- A set of d vectors in V are a basis if they span V.
- A set of d vectors in V are a basis if they are linearly independent.

Why?

- If the d vectors were not independent, then $d-1$ of them would still span V. In the end, we would find a basis of less than d vectors.
- If the d vectors would not span V, then we could add another vector to the set and have $d+1$ independent ones.

Example 7. Are the following sets a basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

No, the set has less than 3 elements.

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

No, the set has more than 3 elements.

Armin Straub
astraub@illinois.edu

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

The set has 3 elements. Hence, it is a basis if and only if the vectors are independent.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Since each column contains a pivot, the three vectors are independent.

Hence, this is a basis of $\mathbb{R}^3$.

Example 8. Let $\mathbb{P}_2$ be the space of polynomials of degree at most 2.

- What is the dimension of $\mathbb{P}_2$?
- Is $\{t, 1-t, 1+t-t^2\}$ a basis of $\mathbb{P}_2$?

Solution.

- The standard basis for $\mathbb{P}_2$ is $\{1, t, t^2\}$.

  This is indeed a basis because every polynomial

  $$a_0 + a_1 t + a_2 t^2$$

  can clearly be written as a linear combination of $1, t, t^2$ in a unique way.

  Hence, $\mathbb{P}_2$ has dimension 3.

- The set $\{t, 1-t, 1+t-t^2\}$ has 3 elements. Hence, it is a basis if and only if the three polynomials are linearly independent.

  We need to check whether

  $$\underbrace{x_1 t + x_2(1-t) + x_3(1+t-t^2)}_{(x_2+x_3)+(x_1-x_2+x_3)t-x_3t^2} = 0$$

  has only the trivial solution $x_1 = x_2 = x_3 = 0$.

  We get the equations

  $$\begin{aligned} x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ -x_3 &= 0 \end{aligned}$$

  which clearly only have the trivial solution. (If you don't see it, solve the system!)

  Hence, $\{t, 1-t, 1+t-t^2\}$ is a basis of $\mathbb{P}_2$.

Armin Straub
astraub@illinois.edu

Shrinking and expanding sets of vectors

We can find a basis for $V = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ by discarding, if necessary, some of the vectors in the spanning set.

Example 9. Produce a basis of $\mathbb{R}^2$ from the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution. Three vectors in $\mathbb{R}^2$ have to be linearly dependent.

Here, we notice that $\boldsymbol{v}_2 = -2\boldsymbol{v}_1$.

The remaining vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$ are a basis of $\mathbb{R}^2$, because the two vectors are clearly independent.

Checking our understanding

Example 10. Subspaces of $\mathbb{R}^3$ can have dimension $0, 1, 2, 3$.

- The only 0-dimensional subspace is $\{\mathbf{0}\}$.
- A 1-dimensional subspace is of the form $\mathrm{span}\{\boldsymbol{v}\}$ where $\boldsymbol{v} \neq 0$.

  These subspaces are lines through the origin.
- A 2-dimensional subspace is of the form $\mathrm{span}\{\boldsymbol{v}, \boldsymbol{w}\}$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ are not multiples of each other.

  These subspaces are planes through the origin.
- The only 3-dimensional subspace is $\mathbb{R}^3$ itself.

True or false?

- Suppose that V has dimension n. Then any set in V containing more than n vectors must be linearly dependent.

  That's correct.
- The space $\mathbb{P}_n$ of polynomials of degree at most n has dimension $n+1$.

  True, as well. A basis is $\{1, t, t^2, \ldots, t^n\}$.
- The vector space of functions $f: \mathbb{R} \to \mathbb{R}$ is infinite-dimensional.

  Yes. A still-infinite-dimensional subspace are the polynomials.
- Consider $V = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$. If one of the vectors, say $\boldsymbol{v}_k$, in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span V.

  True, $\boldsymbol{v}_k$ is not adding anything new.

Armin Straub
astraub@illinois.edu