Midterm!

- Midterm 1: Thursday, 7–8:15pm
  - in 23 Psych if your last name starts with A or B
  - in **Foellinger Auditorium** if your last name starts with C, D, ..., Z
  - bring a picture ID and show it when turning in the exam

Review

- A **vector space** is a set V of vectors which can be added and scaled (without leaving the space!); subject to the "usual" rules.
- $W \subseteq V$ is a **subspace** of V if it is a vector space itself; that is,
  - W contains the zero vector $\mathbf{0}$,
  - W is closed under addition, (i.e. if $\boldsymbol{u}, \boldsymbol{v} \in W$ then $\boldsymbol{u} + \boldsymbol{v} \in W$)
  - W is closed under scaling. (i.e. if $\boldsymbol{u} \in W$ and $c \in \mathbb{R}$ then $c\boldsymbol{u} \in W$)
- $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is always a subspace of V. ($\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are vectors in V)

Example 1. Is $W = \left\{ \begin{bmatrix} 2a-b & 0 \\ b & 3 \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ a subspace of $M_{2\times2}$, the space of 2×2 matrices?

Solution. No, W does not contain the zero "vector".

Example 2. Is $W = \left\{ \begin{bmatrix} 2a-b & 0 \\ b & 3a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ a subspace of $M_{2\times2}$, the space of 2×2 matrices?

Solution. Write "vectors" in W in the form

$$\begin{bmatrix} 2a-b & 0 \\ b & 3a \end{bmatrix} = a \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + b \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

to see that

$$W = \operatorname{span}\left\{ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Like any span, W is a vector space.

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Example 3. Are the following sets vector spaces?

(a) $W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + 3b = 0, 2a - c = 1 \right\}$

No, W_1 does not contain $\mathbf{0}$.

(b) $W_2 = \left\{ \begin{bmatrix} a + c & -2b \\ b + 3c & c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

Yes, $W_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}$.

Hence, W_2 is a subspace of the vector space $\mathrm{Mat}_{2\times 2}$ of all 2×2 matrices.

(c) $W_3 = \left\{ \begin{bmatrix} a + c & -2b \\ b + 3c & c + 7 \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$ (more complicated)

We still have $W_3 = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}$.

Hence, W_3 is a subspace if and only if $\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ is in the span. (We can answer such questions!)

Equivalently (why?!), we have to check whether $\begin{bmatrix} a + c & -2b \\ b + 3c & c + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has solutions a, b, c.

There is no solution ($-2b = 0$ implies $b = 0$, then $b + 3c = 0$ implies $c = 0$; this contradicts $c + 7 = 0$).

(d) $W_4 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : ab \geqslant 0 \right\}$

No. For instance, $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is not in W_4.

(e) W_5 is the set of all polynomials $p(t)$ such that $p'(2) = 1$.

No. W_5 does not contain the zero polynomial.

(f) W_6 is the set of all polynomials $p(t)$ such that $p'(2) = 0$.

Yes. If $p'(2) = 0$ and $q'(2) = 0$, then $(p + q)'(2) = p'(2) + q'(2) = 0$. Likewise for scaling.

Hence, W_6 is a subspace of the vector space of all polynomials.

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What we learned before vector spaces

Linear systems

- Systems of equations can be written as $A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \implies \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

  Sometimes, we represent the system by its augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

- A linear system has either
  - no solution (such a system is called **inconsistent**),

    $\iff$ echelon form contains row $[\ 0\ \ \ldots\ \ 0 \mid b\ ]$ with $b \neq 0$
  - one unique solution,

    $\iff$ system is consistent and has no free variables
  - infinitely many solutions.

    $\iff$ system is consistent and has at least one free variable
- We know different techniques for solving systems $A\boldsymbol{x} = \boldsymbol{b}$.
  - Gaussian elimination on $[\ A\ \ \boldsymbol{b}\ ]$
  - LU decomposition $A = LU$
  - using matrix inverse, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

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Matrices and vectors

- A **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is of the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m.$$

- $\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all such linear combinations.
  - Spans are always vector spaces.
  - For instance, a span in $\mathbb{R}^3$ can be $\{\boldsymbol{0}\}$, a line, a plane, or $\mathbb{R}^3$.
- The **transpose** A^T of a matrix A has rows and columns flipped.

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

  - $(A+B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$
- An $m \times n$ **matrix** A has m rows and n columns.
- The product $A\boldsymbol{x}$ of **matrix times vector** is

$$\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_n\boldsymbol{a}_n.$$

- Different interpretations of the product of **matrix times matrix**:
  - column interpretation

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3c & b & c \\ d+3f & e & f \\ g+3i & h & i \end{bmatrix}$$

  - row interpretation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

  - row-column rule

$$(AB)_{i,j} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

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- The **inverse** A^{-1} of A is characterized by $A^{-1}A = I$ (or $AA^{-1} = I$).
  - $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
  - Can compute A^{-1} using Gauss–Jordan method.

$$\begin{bmatrix} A & I \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

  - $(A^T)^{-1} = (A^{-1})^T$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - An $n \times n$ matrix A is invertible

    $\Longleftrightarrow$ A has n pivots

    $\Longleftrightarrow$ $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution (if true for one $\boldsymbol{b}$, then true for all $\boldsymbol{b}$)

Gaussian elimination

- **Gaussian elimination** can bring any matrix into an **echelon form**.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  It proceeds by **elementary row operations**:
  - **(replacement)** Add one row to a multiple of another row.
  - **(interchange)** Interchange two rows.
  - **(scaling)** Multiply all entries in a row by a nonzero constant.
- Each elementary row operation can be encoded as multiplication with an **elementary matrix**.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e-a & f-b & g-c & h-d \\ i & j & k & l \end{bmatrix}$$

- We can continue row reduction to obtain the (unique) RREF.

Using Gaussian elimination

Gaussian elimination and row reductions allow us:

- solve systems of linear systems

$$\left[\begin{array}{cccc|c} 0 & 3 & -6 & 4 & -5 \\ 3 & -7 & 8 & 8 & 9 \\ 3 & -9 & 12 & 6 & 15 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -24 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \begin{cases} x_1 = -24 + 2x_3 \\ x_2 = -7 + 2x_3 \\ x_3 \text{ free} \\ x_4 = 4 \end{cases}$$

- compute the LU decomposition $A = LU$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix}$$

- compute the inverse of a matrix

to find $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$, we use Gauss–Jordan:

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

- determine whether a vector is a linear combination of other vectors

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ if and only if

the system corresponding to $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 3 \end{array}\right]$ is consistent.

(Each solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ gives a linear combination $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.)