Review

- The **inverse** A^{-1} of a matrix A is, if it exists, characterized by

$$AA^{-1} = A^{-1}A = I_n.$$

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- If A is invertible, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.
- Gauss–Jordan method to compute A^{-1}:
  - bring to RREF $[\ A \mid I\ ] \rightsquigarrow [\ I \mid A^{-1}\ ]$
- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$

  Why? Because $(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B = I$

Further properties of matrix inverses

Theorem 1. Let A be an $n \times n$ matrix. Then the following statements are equivalent: (i.e., for a given A, they are either all true or all false)

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivots. (Easy to check!)

(d) For every $\boldsymbol{b}$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

Namely, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

(e) There is a matrix B such that $AB = I_n$. (A has a "right inverse".)

(f) There is a matrix C such that $CA = I_n$. (A has a "left inverse".)

Note. Matrices that are not invertible are often called **singular**.

The book uses **singular** for $n \times n$ matrices that do not have n pivots. As we just saw, it doesn't make a difference.

Example 2. We now see at once that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Why? Because it has only one pivot.

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Application: finite differences

Let us apply linear algebra to the **boundary value problem** (BVP)

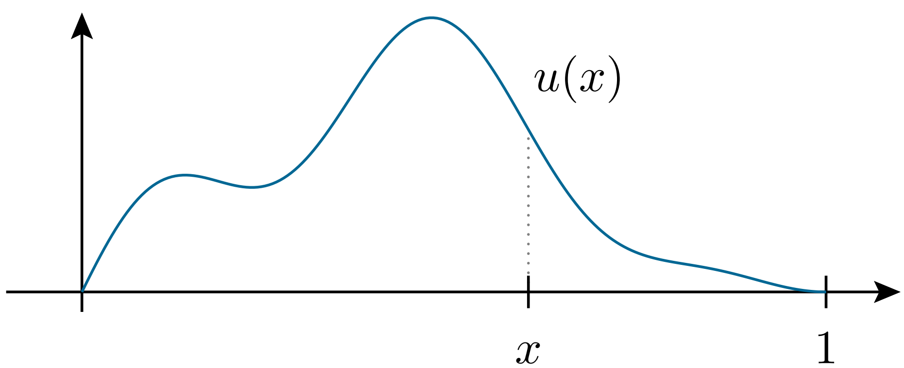
$$-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f(x), \quad 0 \leqslant x \leqslant 1, \qquad u(0) = u(1) = 0.$$

$f(x)$ is given, and the goal is to find $u(x)$.

Physical interpretation: models steady-state temperature distribution in a bar ($u(x)$ is temperature at point x) under influence of an external heat source $f(x)$ and with ends fixed at $0°$ (ice cube at the ends?).

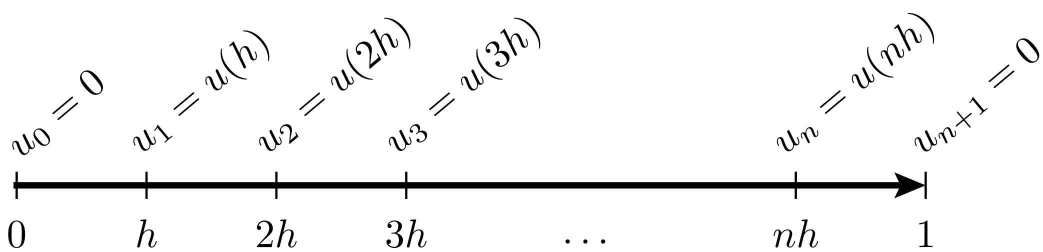
Remark 3. Note that this simple BVP can be solved by integrating $f(x)$ twice. We get two constants of integration, and so we see that the boundary condition $u(0) = u(1) = 0$ makes the solution $u(x)$ unique.

Of course, in the real applications the BVP would be harder. Also, $f(x)$ might only be known at some points, so we cannot use calculus to integrate it.



We will approximate this problem as follows:

- replace $u(x)$ by its values at equally spaced points in $[0, 1]$



- approximate $\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$ at these points (**finite differences**)
- replace differential equation with linear equation at each point
- solve linear problem using Gaussian elimination

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Finite differences

Finite differences for first derivative:

$$\frac{\mathrm{d}u}{\mathrm{d}x} \approx \frac{\Delta u}{\Delta x} = \frac{u(x+h) - u(x)}{h} \overset{\text{or}}{=} \frac{u(x) - u(x-h)}{h} \overset{\text{or}}{=} \underbrace{\frac{u(x+h) - u(x-h)}{2h}}_{\text{symmetric and most accurate}}$$

Note. Recall that you can always use L'Hospital's rule to determine the limit of such quantities (especially more complicated ones) as $h \rightarrow 0$.

Finite difference for second derivative:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \approx \underbrace{\frac{u(x+h) - 2u(x) + u(x-h)}{h^2}}_{\text{the only symmetric choice involving only } u(x),\ u(x \pm h)}$$

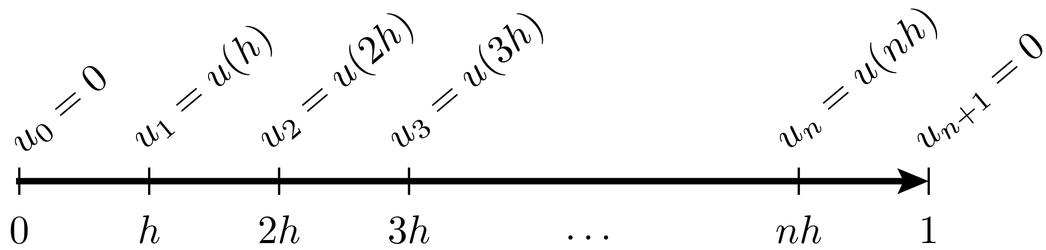
Question 4. Why does this approximate $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$ as $h \rightarrow 0$?

Solution.

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \approx \frac{\frac{\mathrm{d}u}{\mathrm{d}x}(x+h) - \frac{\mathrm{d}u}{\mathrm{d}x}(x)}{h} \approx \frac{\frac{u(x+h) - u(x)}{h} - \frac{u(x) - u(x-h)}{h}}{h} \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Setting up the linear equations

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x), \quad 0 \leqslant x \leqslant 1, \qquad u(0) = u(1) = 0.$$



Using $-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \approx -\frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$, we get:

at $x = h$: $-\frac{u(2h) - 2u(h) + u(0)}{h^2} = f(h)$

$$\Longrightarrow \quad 2u_1 - u_2 = h^2 f(h) \qquad (1)$$

at $x = 2h$: $-\frac{u(3h) - 2u(2h) + u(h)}{h^2} = f(2h)$

$$\Longrightarrow \quad -u_1 + 2u_2 - u_3 = h^2 f(2h) \qquad (2)$$

at $x = 3h$:

$$\Longrightarrow \quad -u_2 + 2u_3 - u_4 = h^2 f(3h) \qquad (3)$$

$$\vdots$$

at $x = nh$: $-\frac{u((n+1)h) - 2u(nh) + u((n-1)h)}{h^2} = f(nh)$

$$\Longrightarrow \quad -u_{n-1} + 2u_n = h^2 f(nh) \qquad (n)$$

Example 5. In the case of six divisions ($n = 5$, $h = \frac{1}{6}$), we get:

$$\underbrace{\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} h^2 f(h) \\ h^2 f(2h) \\ h^2 f(3h) \\ h^2 f(4h) \\ h^2 f(5h) \end{bmatrix}}_{\boldsymbol{b}}$$

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Such a matrix is called a **band matrix**. As we will see next, such matrices always have a particularly simple LU decomposition.

Gaussian elimination:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \underset{\begin{bmatrix} 1 & & & & \\ \frac{1}{2} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}{\overset{R2 \to R2 + \frac{1}{2}R1}{\rightsquigarrow}} \begin{bmatrix} 2 & -1 & & & \\ 0 & \frac{3}{2} & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$$\underset{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & \frac{2}{3} & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}{\overset{R3 \to R3 + \frac{2}{3}R2}{\rightsquigarrow}} \begin{bmatrix} 2 & -1 & & & \\ 0 & \frac{3}{2} & -1 & & \\ & 0 & \frac{4}{3} & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$$\underset{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & \frac{3}{4} & 1 & \\ & & & & 1 \end{bmatrix}}{\overset{R4 \to R4 + \frac{3}{4}R3}{\rightsquigarrow}} \begin{bmatrix} 2 & -1 & & & \\ 0 & \frac{3}{2} & -1 & & \\ & 0 & \frac{4}{3} & -1 & \\ & & 0 & \frac{5}{4} & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$$\underset{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \frac{4}{5} & 1 \end{bmatrix}}{\overset{R5 \to R5 + \frac{4}{5}R4}{\rightsquigarrow}} \begin{bmatrix} 2 & -1 & & & \\ 0 & \frac{3}{2} & -1 & & \\ & 0 & \frac{4}{3} & -1 & \\ & & 0 & \frac{5}{4} & -1 \\ & & & 0 & \frac{6}{5} \end{bmatrix}$$

In conclusion, we have the LU decomposition:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & -1 & \\ & & & \frac{5}{4} & -1 \\ & & & & \frac{6}{5} \end{bmatrix}$$

That's how the LU decomposition of band matrices always looks like.

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