Review

Example 1. Elementary matrices in action:

(a) $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 7g & 7h & 7i \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

(d) $$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3c & b & c \\ d+3f & e & f \\ g+3i & h & i \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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LU decomposition, continued

Gaussian elimination revisited

Example 2. Keeping track of the elementary matrices during Gaussian elimination on A:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} \quad R2 \to R2 - 2R1$$

$$EA = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

Note that:

$$A = E^{-1} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

We factored A as the product of a lower and upper triangular matrix!

We say that A has **triangular factorization**.

$A = LU$ is known as the **LU decomposition** of A.

L is lower triangular, U is upper triangular.

Definition 3.

lower triangular

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & * & \cdots & * \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}$$

missing entries are 0

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Example 4. Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ as $A = LU$.

Solution. We begin with $R2 \to R2 - 2R1$ followed by $R3 \to R3 + R1$:

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The factor L is given by: note that $E_3 E_2 E_1 A = U \implies A = E_1^{-1} E_2^{-1} E_3^{-1} U$

$$\begin{aligned} L &= E_1^{-1} E_2^{-1} E_3^{-1} \\ &= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

In conclusion, we found the following LU decomposition of A:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix}$$

Note: The extra steps to compute L were unnecessary! The entries in L are precisely the negatives of the ones in the elementary matrices during elimination. Can you see it?

Once we have $A = LU$, it is simple to solve $A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{aligned} & A\boldsymbol{x} = \boldsymbol{b} \\ \iff\ & L(U\boldsymbol{x}) = \boldsymbol{b} \\ \iff\ & L\boldsymbol{c} = \boldsymbol{b} \quad \text{and} \quad U\boldsymbol{x} = \boldsymbol{c} \end{aligned}$$

Both of the final systems are triangular and hence easily solved:

- $L\boldsymbol{c} = \boldsymbol{b}$ by forward substitution to find $\boldsymbol{c}$, and then
- $U\boldsymbol{x} = \boldsymbol{c}$ by backward substitution to find $\boldsymbol{x}$.

Important practical point: can be quickly repeated for many different $\boldsymbol{b}$.

Example 5. Solve $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}$.

Solution. We already found the LU decomposition $A = LU$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix}$$

Forward substitution to solve $L\boldsymbol{c} = \boldsymbol{b}$ for $\boldsymbol{c}$:

$$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{c} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

Backward substitution to solve $U\boldsymbol{x} = \boldsymbol{c}$ for $\boldsymbol{x}$:

$$\begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

It's always a good idea to do a quick check:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}$$

Triangular factors for any matrix

Can we factor any matrix A as $A = LU$?

Yes, almost! Think about the process of Gaussian elimination.

- In each step, we use a pivot to produce zeros below it.
  The corresponding elementary matrices are lower diagonal!
- The only other thing we might have to do, is a row exchange.
  Namely, if we run into a zero in the position of the pivot.
- All of these row exchanges can be done at the beginning!

Definition 6. A **permutation matrix** is one that is obtained by performing row exchanges on an identity matrix.

Example 7. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is a permutation matrix.

EA is the matrix obtained from A by permuting the last two rows.

Theorem 8. For any matrix A there is a permutation matrix P such that $PA = LU$.

In other words, it might not be possible to write A as $A = LU$, but we only need to permute the rows of A and the resulting matrix PA now has an LU decomposition: $PA = LU$.

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Practice problems

- Is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ upper triangular? Lower triangular?
- Is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ upper triangular? Lower triangular?
- True or false?
  - A permutation matrix is one that is obtained by performing column exchanges on an identity matrix.
- Why do we care about LU decomposition if we already have Gaussian elimination?

Example 9. Solve $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$ using the factorization we already have.

Example 10. The matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

cannot be written as $A = LU$ (so it doesn't have a LU decomposition). But there is a permutation matrix P such that PA has a LU decomposition.

Namely, let $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then $PA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

PA can now be factored as $PA = LU$. Do it!!

(By the way, $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ would work as well.)