Review

- We have a standardized recipe to find all solutions of systems such as:

$$\begin{array}{rrrrrcl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \end{array}$$

- The computational part is to start with the **augmented matrix**

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right],$$

and to calculate its **reduced echelon form** (which is unique!). Here:

$$\left[\begin{array}{rrrrr|r} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

- **pivot variables** (or **basic variables**): x_1, x_2, x_5

  free variables: x_3, x_4
- solving each equation for the pivot variables in terms of the free variables:

$$\begin{array}{rrrrcr} x_1 & & -2x_3 & +3x_4 & & = -24 \\ & x_2 & -2x_3 & +2x_4 & & = -7 \\ & & & & x_5 & = 4 \end{array} \qquad \begin{cases} x_1 = -24 + 2x_3 - 3x_4 \\ x_2 = -7 + 2x_3 - 2x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \\ x_5 = 4 \end{cases}$$

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Questions of existence and uniqueness

The question whether a system has a solution and whether it is unique, is easier to answer than to determine the solution set.

All we need is an echelon form of the augmented matrix.

Example 1. Is the following system consistent? If so, does it have a unique solution?

$$\begin{array}{rrrrrcl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \end{array}$$

Solution. In the course of an earlier example, we obtained the echelon form:

$$\left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Hence, it is consistent (imagine doing back-substitution to get a solution).

Theorem 2. (Existence and uniqueness theorem) A linear system is **consistent** if and only if an echelon form of the augmented matrix has **no** row of the form

$$[\,0\ \ \ldots\ \ 0\,|\,b\,],$$

where b is nonzero.

If a linear system is consistent, then the solutions consist of either

- a unique solution (when there are no free variables) or
- infinitely many solutions (when there is at least one free variable).

Example 3. For what values of h will the following system be consistent?

$$\begin{array}{rrcl} 3x_1 & -9x_2 & = & 4 \\ -2x_1 & +6x_2 & = & h \end{array}$$

Solution. We perform row reduction to find an echelon form:

$$\left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \overset{R2 \to R2 + \frac{2}{3}R1}{\rightsquigarrow} \left[\begin{array}{rr|r} 3 & -9 & 4 \\ 0 & 0 & h + \frac{8}{3} \end{array}\right]$$

The system is consistent if and only if $h = -\frac{8}{3}$.

Brief summary of what we learned so far

- Each linear system corresponds to an augmented matrix.
- Using Gaussian elimination (i.e. row reduction to echelon form) on the augmented matrix of a linear system, we can
  - read off, whether the system has no, one, or infinitely many solutions;
  - find all solutions by back-substitution.
- We can continue row reduction to the reduced echelon form.
  - Solutions to the linear system can now be just read off.
  - This form is unique!

Note. Besides for solving linear systems, Gaussian elimination has other important uses, such as computing determinants or inverses of matrices.

A recipe to solve linear systems (Gauss–Jordan elimination)

(1) Write the augmented matrix of the system.

(2) Row reduce to obtain an equivalent augmented matrix in echelon form.
Decide whether the system is consistent. If not, stop; otherwise go to the next step.

(3) Continue row reduction to obtain the reduced echelon form.

(4) Express this final matrix as a system of equations.

(5) Declare the free variables and state the solution in terms of these.

Questions to check our understanding

- On an exam, you are asked to find all solutions to a system of linear equations. You find exactly two solutions. Should you be worried?

  Yes, because if there is more than one solution, there have to be infinitely many solutions. Can you see how, given two solutions, one can construct infinitely many more?

- True or false?
  - There is no more than one pivot in any row.

    True, because a pivot is the first nonzero entry in a row.

  - There is no more than one pivot in any column.

    True, because in echelon form (that's where pivots are defined) the entries below a pivot have to zero.

  - There cannot be more free variables than pivot variables.

    False, consider, for instance, the augmented matrix $\left[\begin{array}{ccc|c} 1 & 7 & 5 & 3 \end{array}\right]$.

The geometry of linear equations

Adding and scaling vectors

Example 4. We have already encountered **matrices** such as

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & -1 & 2 & 2 \\ 3 & 2 & -2 & 0 \end{bmatrix}.$$

Each column is what we call a **(column) vector**.

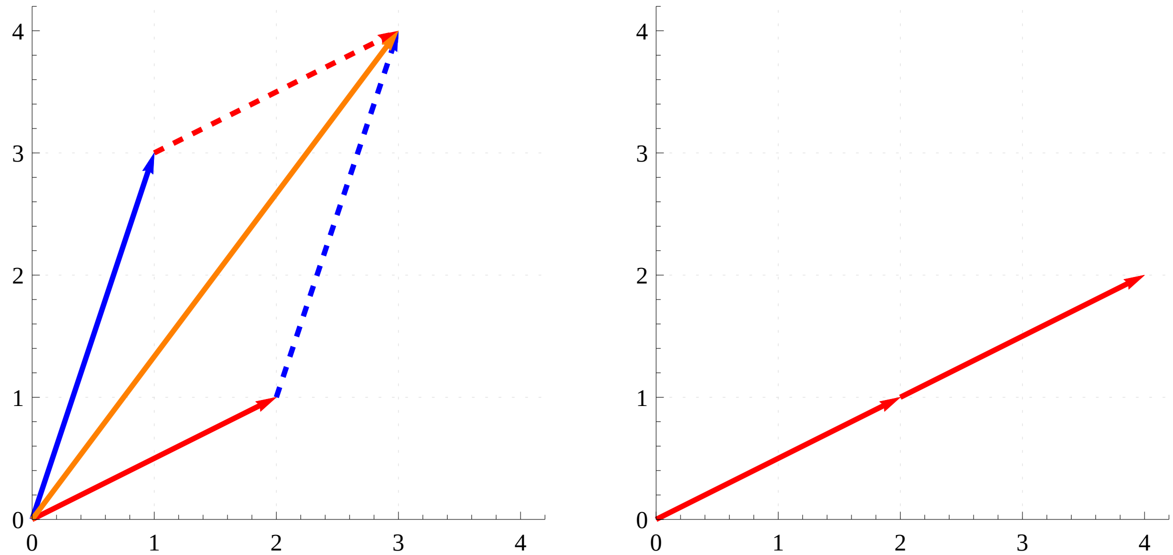
In this example, each column vector has 3 entries and so lies in $\mathbb{R}^3$.

Example 5. A fundamental property of vectors is that vectors of the same kind can be **added** and **scaled**.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}, \qquad 7 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \\ 7x_3 \end{bmatrix}.$$

Example 6. (Geometric description of $\mathbb{R}^2$) A vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ represents the point (x_1, x_2) in the plane.

Given $\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, graph $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{x} + \boldsymbol{y}$, $2\boldsymbol{y}$.



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Adding and scaling vectors, the most general thing we can do is:

Definition 7. Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_m$, the vector

$$c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \ldots + c_m \boldsymbol{v}_m$$

is a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

The scalars $c_1, \ldots, c_m$ are the **coefficients** or **weights**.

Example 8. Linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ include:

- $3\boldsymbol{v}_1 - \boldsymbol{v}_2 + 7\boldsymbol{v}_3$,
- $\boldsymbol{v}_2 + \boldsymbol{v}_3$,
- $\frac{1}{3}\boldsymbol{v}_2$,
- $\boldsymbol{0}$.

Example 9. Express $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solution. We have to find c_1 and c_2 such that

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

This is the same as:

$$\begin{array}{rrcl} 2c_1 & -c_2 & = & 1 \\ c_1 & +c_2 & = & 5 \end{array}$$

Solving, we find $c_1 = 2$ and $c_2 = 3$.

Indeed,

$$2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Note that the augmented matrix of the linear system is

$$\left[\begin{array}{rr|r} 2 & -1 & 1 \\ 1 & 1 & 5 \end{array}\right],$$

and that this example provides a new way of think about this system.

The row and column picture

Example 10. We can think of the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

in two different geometric ways.

Row picture.

Each equation defines a line in $\mathbb{R}^2$.

Which points lie on the intersection of these lines?

Column picture.

The system can be written as $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Which linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produce $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?

This example has the unique solution $x = 2$, $y = 3$.

- $(2, 3)$ is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.
- $2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the (only) linear combination producing $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

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