

Midterm #2

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 31 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points) In the case $\lambda > 0$, find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0.$$

Solution. Since $\lambda > 0$, the characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A \cos(rx) + B \sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A \cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n . Since $r > 0$, we have $n \geq 0$.

Hence, the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \dots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Problem 2. (3 points) Let $y(x)$ be the unique solution to the IVP $y'' = 5 + 2(x-1)y^2$, $y(0) = 1$, $y'(0) = 2$.

Determine the first several terms (up to x^3) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 5 + 2 \cdot (-1) \cdot y(0)^2 = 3$.

Differentiating both sides of the DE, we obtain $y''' = 2y^2 + 4(x-1)yy'$.

In particular, $y'''(0) = 2y(0)^2 + 4 \cdot (-1) \cdot y(0) \cdot y'(0) = -6$.

Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \dots = 1 + 2x + \frac{3}{2}x^2 - x^3 + \dots$

Problem 3. (6 points) Derive a recursive description of a power series solution $y(x)$ of the DE $y'' = (5x^2 - 3)y$.

Solution. Let us spell out the power series for y, x^2y, y'' :

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad x^2y(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

Hence, the DE becomes:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = 5 \sum_{n=2}^{\infty} a_{n-2} x^n - 3 \sum_{n=0}^{\infty} a_n x^n.$$

We compare coefficients of x^n :

- $n = 0$: $2a_2 = -3a_0$, so that $a_2 = -\frac{3}{2}a_0$.
- $n = 1$: $6a_3 = -3a_1$, so that $a_3 = -\frac{1}{2}a_1$.
- $n \geq 2$: $(n+2)(n+1)a_{n+2} = 5a_{n-2} - 3a_n$

Equivalently, for $n \geq 4$, $a_n = \frac{-3}{n(n-1)}a_{n-2} + \frac{5}{n(n-1)}a_{n-4}$.

In conclusion, the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is recursively determined by

$$a_2 = -\frac{3}{2}a_0, \quad a_3 = -\frac{1}{2}a_1, \quad a_n = \frac{-3}{n(n-1)}a_{n-2} + \frac{5}{n(n-1)}a_{n-4} \quad \text{for } n \geq 4.$$

(The values a_0 and a_1 are the initial conditions.)

Problem 4. (3 points) A mass-spring system is described by the DE $3y'' + ky = F(t)$ where $F(t)$ is an external force with period 5. For which values of k can resonance occur?

Solution. $F(t)$ has a Fourier series of the form $F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{5}\right) + b_n \sin\left(\frac{2\pi nt}{5}\right) \right)$.

The roots of $p(D) = 3D^2 + k$ are $\pm i\sqrt{\frac{k}{3}}$, so that the natural frequency is $\sqrt{\frac{k}{3}}$. Resonance therefore can occur if $\sqrt{\frac{k}{3}} = \frac{2\pi n}{5}$ for some $n \in \{1, 2, 3, \dots\}$. Equivalently, resonance can occur if $k = \frac{12\pi^2 n^2}{25}$ for some $n \in \{1, 2, 3, \dots\}$.

Problem 5. (3 points) Find a minimum value for the radius of convergence of a power series solution to

$$(x+3)y'' + 2y = \frac{\cos(x)}{x^2+4} \quad \text{at } x=1.$$

Solution. Note that this is a linear DE! (Otherwise, we could not proceed.) Rewriting the DE as $y'' + \frac{2}{x+3}y = \frac{\cos(x)}{(x^2+4)(x+3)}$, we see that the singular points are $x = \pm 2i, -3$.

Note that $x=1$ is an ordinary point of the DE and that the distance to the nearest singular point is $|1 - (\pm 2i)| = \sqrt{1^2 + 2^2} = \sqrt{5}$ (the distance to -3 is $|1 - (-3)| = 4 > \sqrt{5}$).

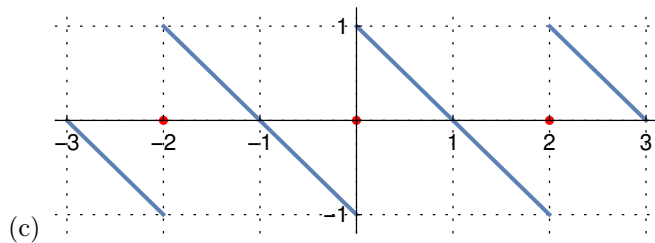
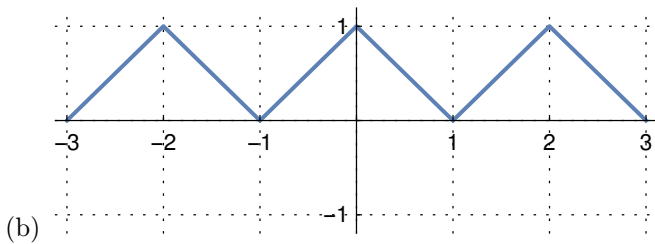
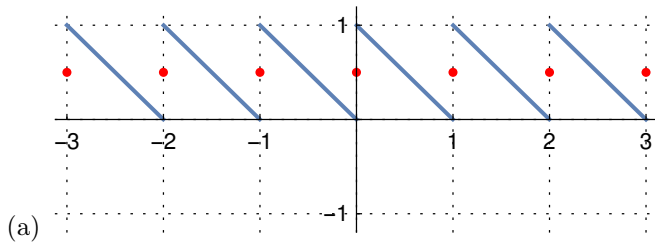
Hence, the DE has power series solutions about $x=1$ with radius of convergence at least $\sqrt{5}$.

Problem 6. (4 points) Consider the function $f(t) = 1 - t$, defined for $t \in [0, 1]$.

- (a) Fourier series of $f(t)$ (b) Fourier cosine series of $f(t)$ (c) Fourier sine series of $f(t)$

In each sketch, carefully mark the values of the Fourier series at discontinuities.

Solution.



Problem 7. (6 points)

- (a) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + 5y = 1 + 2xe^{3x}\sinh(4x)$. Using the operator D write down a homogeneous linear differential equation which y_p solves.

- (b) Determine the power series around $x = 0$: $\frac{1}{1+x^2} =$

- (c) Determine the power series around $x = 0$: $2\cosh(3x) =$

- (d) Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x+2)^n$. How can we compute the a_n from $y(x)$? $a_n =$.

(e) If $f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{3}\right)$, then we can compute the b_n as $b_n =$.

Solution.

(a) $(D+1)^2(D-7)^2D(D^2+5)y=0$

(b) Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

(c) Since $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ (even part of e^x), we have $2\cosh(3x) = 2 \sum_{n=0}^{\infty} \frac{3^{2n} x^{2n}}{(2n)!}$.

(d) $a_n = \frac{y^{(n)}(-2)}{n!}$ because this is the Taylor series of $f(x)$ around $x = -2$.

(e) This is the Fourier series of $f(t)$ (which has period 6 and so is $2L$ -periodic with $L=3$) and it has the extra property that the a_n coefficients happen to be zero. The Fourier coefficients b_n can be computed as

$$b_n = \frac{1}{3} \int_{-3}^3 f(t) \sin\left(\frac{n\pi t}{3}\right) dt.$$

(extra scratch paper)