Midterm #1

MATH 332 — Differential Equations II
Monday, Feb 23, 2026

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 33 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (8 points) Let $M = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

(a) Compute e^{Mt}.

(b) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}\right) = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

Hence, the eigenvalues are $\lambda = 2$ and $\lambda = 3$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 2$, we need to solve $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 3$, we need to solve $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.

Hence, a fundamental matrix solution is $\Phi(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}$.

Note that $\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, so that $\Phi(0)^{-1} = \frac{1}{2-1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. It follows that

$$e^{Mt} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{2t} & e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} - e^{3t} & -e^{2t} + e^{3t} \\ 2e^{2t} - 2e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}.$$

(b) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mt}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{2t} - e^{3t} \\ 2e^{2t} - 2e^{3t} \end{bmatrix}$.

Problem 2. (2 points) If $M^n = \begin{bmatrix} 2 - 4^n & -2 + 2 \cdot 4^n \\ 1 - 4^n & -1 + 2 \cdot 4^n \end{bmatrix}$, then $e^{Mx} =$ $\boxed{}$.

Solution. $e^{Mx} = \begin{bmatrix} 2e^x - e^{4x} & -2e^x + 2e^{4x} \\ e^x - e^{4x} & -e^x + 2e^{4x} \end{bmatrix}$

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Problem 3. (6 points) Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 5$, $a_1 = -5$.

(a) Find an explicit (Binet-like) formula for a_n.

(b) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6 = (N+2)(N-3)$ has roots $3, -2$.

Hence, $a_n = C_1 3^n + C_2 (-2)^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $a_0 = C_1 + C_2 = 5$, $a_1 = 3C_1 - 2C_2 = -5$.

Solving, we find $C_1 = 1$ and $C_2 = 4$ so that, in conclusion, $a_n = 3^n + 4(-2)^n$.

(b) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$.

Problem 4. (6 points) Fill in the blanks. None of the problems should require any computation!

(a) Consider a homogeneous linear differential equation with constant real coefficients which has order 5. Suppose $y(x) = 7 + 2x e^{3x} \cos(4x)$ is a solution. Write down the general solution.

(b) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + 5y = 4x^2 e^x$. Find a homogeneous linear differential equation which y_p solves.

You can use the operator D to write the DE. No need to simplify, any form is acceptable.

(c) Determine a (homogeneous linear) recurrence equation satisfied by $a_n = 4 - (2n+1)3^n$.

You can use the operator N to write the recurrence. No need to simplify, any form is acceptable.

Solution.

(a) $y(x) = C_1 + (C_2 + C_3 x)e^{3x}\cos(4x) + (C_4 + C_5 x)e^{3x}\sin(4x)$.

Explanation. $y(x) = 7 + 2x e^{3x}\cos(4x)$ is a solution of $p(D)y = 0$ if and only if $0, 3 \pm 4i, 3 \pm 4i$ are roots of the characteristic polynomial $p(D)$. Since the order of the DE is 5, there can be no further roots.

(b) $(D-1)^3(D^2+5)y = 0$

Explanation. Since y_p solves the inhomogeneous DE, we have $(D^2+5)y_p = 4x^2 e^x$. The right-hand side is a solution of $p(D)y = 0$ if and only if $1, 1, 1$ are roots of the characteristic polynomial $p(D)$. In particular, $(D-1)^3 4x^2 e^x = 0$. Combined, we find that $(D-1)^3(D^2+5)y_p = 0$.

(c) $(N-3)^2(N-1)a_n = 0$

Explanation. $a_n = 4 \cdot 1^n - (2n+1)3^n$ is a solution of $p(N)a_n = 0$ if and only if $1, 3, 3$ are among the roots of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-3)^2(N-1)$.

[Since, $(N-4)^2(N-1) = N^3 - 7N^2 + 15N - 9$, the recurrence in explicit form is $a_{n+3} = 7a_{n+2} - 15a_{n+1} + 9a_n$.]

Problem 5. (8 points) Determine all equilibrium points of the system $\frac{dx}{dt} = (x-2)y$, $\frac{dy}{dt} = y - x$. Classify the stability and type of each equilibrium point.

Solution. We solve $(x-2)y = 0$ and $y - x = 0$. The first equation implies that $x = 2$ or $y = 0$. Combined with the second equation, which implies $x = y$, we find that the equilibrium points are $(2,2)$ and $(0,0)$.

Our system is $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ with $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} (x-2)y \\ y-x \end{bmatrix}$.

The Jacobian matrix is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y & x-2 \\ -1 & 1 \end{bmatrix}$.

- At $(2,2)$, the Jacobian matrix is $\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$. The eigenvalues are $2, 1$ (since this is a triangular matrix, we can just read this off). Since both are positive, $(2,2)$ is a nodal source. In particular, $(2,2)$ is unstable.

- At $(0,0)$, the Jacobian matrix is $\begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & -2 \\ -1 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$ so that the eigenvalues are $2, -1$. Since one is positive and the other is negative, $(0,0)$ is a saddle. In particular, $(0,0)$ is unstable.

Problem 6. (3 points) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' + 2y_1 &= 5y_2 \\ y_2'' + 4y_2 &= 7y_1' \end{aligned} \qquad y_1(0) = 1,\ y_1'(0) = -2,\ y_2(0) = 3,\ y_2'(0) = 0$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 5 & 0 & 0 \\ 0 & -4 & 7 & 0 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}.$$

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(extra scratch paper)