

# Midterm #1 – Practice

Please print your name:

---

**Bonus challenge.** Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

**Reminder.** No notes, calculators or tools of any kind will be permitted on the midterm exam.

**Problem 1.** Let  $M = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix}$ .

- (a) Determine the general solution to  $\mathbf{y}' = M\mathbf{y}$ .
- (b) Determine a fundamental matrix solution to  $\mathbf{y}' = M\mathbf{y}$ .
- (c) Compute  $e^{Mx}$ .
- (d) Solve the initial value problem  $\mathbf{y}' = M\mathbf{y}$  with  $\mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .
- (e) Determine all equilibrium points of  $\mathbf{y}' = M\mathbf{y}$  as well as their stability and type.

**Problem 2.**

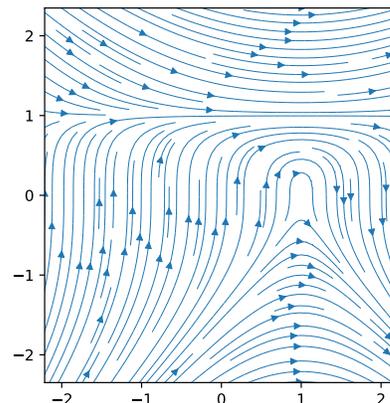
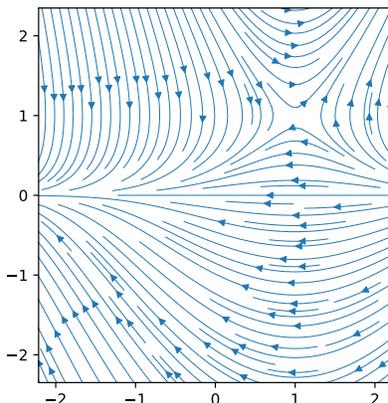
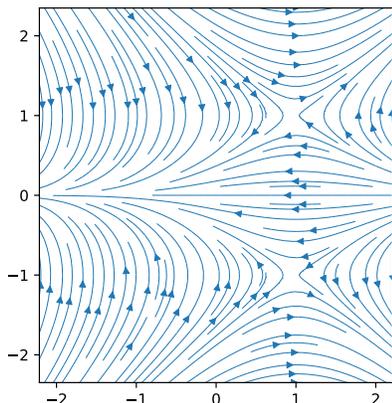
- (a) Write the differential equation  $y'''' + 7y'' - 3y' + y = 0$  as a system of (first-order) differential equations.
- (b) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 3y_1' + 2y_2' - 5y_1 & y_1(0) &= 1, \quad y_1'(0) = -2, \quad y_2(0) = 3, \quad y_2'(0) = 0 \\ y_2'' &= y_1' - y_2' + 3y_2 \end{aligned}$$

Write it as a first-order initial value problem in the form  $\mathbf{y}' = M\mathbf{y}$ ,  $\mathbf{y}(0) = \mathbf{y}_0$ .

**Problem 3.**

- (a) Circle the phase portrait below which belongs to  $\frac{dx}{dt} = y^2 - 1$ ,  $\frac{dy}{dt} = y \cdot (x - 1)$ .
- (b) Determine all equilibrium points as well as their stability and type.



**Problem 4.**

- (a) Write down a (homogeneous linear) recurrence equation satisfied by  $a_n = 3^n - 2^n$ .
- (b) Write down a (homogeneous linear) recurrence equation satisfied by  $a_n = n^2 3^n - 2^n$ .

**Problem 5.** Consider the sequence  $a_n$  defined by  $a_{n+2} = a_{n+1} + 6a_n$  and  $a_0 = 3$ ,  $a_1 = -1$ .

- (a) Determine the first few terms of the sequence.
- (b) Find a Binet-like formula for  $a_n$ .
- (c) Determine  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ .

**Problem 6.**

- (a) Find the general solution to  $y^{(5)} - 4y^{(4)} + 5y''' - 2y'' = 0$ .
- (b) Find the general solution to  $y''' - y = e^x + 7$ .
- (c) Solve  $y'' + 2y' + y = 2e^{2x} + e^{-x}$ ,  $y(0) = -1$ ,  $y'(0) = 2$ .
- (d) Find the general solution to  $y'' - 4y' + 4y = 3e^{2x}$ .
- (e) Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose  $y(x) = x^2 e^{2x} \cos(x)$  is a solution. Write down the general solution.
- (f) Write down a homogeneous linear differential equation satisfied by  $y(x) = 1 - 5x^2 e^{-2x}$ .
- (g) Let  $y_p$  be any solution to the inhomogeneous linear differential equation  $y'' + xy = e^x$ . Find a homogeneous linear differential equation which  $y_p$  solves. *Hint: Do not attempt to solve the DE.*

**Problem 7.** Let  $M = \begin{bmatrix} 1 & 4 \\ 6 & -1 \end{bmatrix}$ .

- (a) Determine the general solution to  $\mathbf{a}_{n+1} = M\mathbf{a}_n$ .
- (b) Determine a fundamental matrix solution to  $\mathbf{a}_{n+1} = M\mathbf{a}_n$ .
- (c) Compute  $M^n$ .
- (d) Without further computations, determine  $e^{Mt}$ .
- (e) Determine all equilibrium points of  $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$  as well as their stability and type.