- Harmonic functions, maximum principle, ...
- Real and imaginary parts of analytic functions are harmonic.
- Lorenz system, chaotic behaviour, strange attractors, ...
  https://en.wikipedia.org/wiki/Lorenz_system

Excursion: The Riemann hypothesis—A Millennium Prize Problem

The **Riemann zeta function** is defined by $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$.

Note that this series converges (for real s) if and only if $s > 1$.

The divergent series $\zeta(1)$ is the harmonic series, and $\zeta(p)$ is often called a p-series in Calculus II.

Example 179. Recall from Example 140 that using Fourier series, we were able to find that $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ Use this to derive $\zeta(2) = \sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution. If we split the sum into those terms where n is even and those where n is odd, then we get

$$\sum_{n \geqslant 1} \frac{1}{n^2} = \sum_{n \geqslant 1} \frac{1}{(2n)^2} + \sum_{n \geqslant 1} \frac{1}{(2n-1)^2} = \frac{1}{4} \sum_{n \geqslant 1} \frac{1}{n^2} + \frac{\pi^2}{8}.$$

If we write $x = \sum_{n \geqslant 1} \frac{1}{n^2}$, then this means that $x = \frac{1}{4} x + \frac{\pi^2}{8}$. We can then solve this equation to find $x = \frac{\pi^2}{6}$, which is what we wanted to derive.

Comment. Euler achieved worldwide fame in 1734 by discovering and proving that $\zeta(2) = \frac{\pi^2}{6}$ (as well as $\sum_{n \geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}$ and similar formulas for $\zeta(6), \zeta(8), \ldots$).

Comment. On the other hand, no such evaluations are known for $\zeta(3), \zeta(5), \ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

The Clay Mathematics Institute has offered 10^6 dollars each for the first correct solution to seven **Millennium Prize Problems**. Six of the seven problems remain open.

https://en.wikipedia.org/wiki/Millennium_Prize_Problems

Comment. Grigori Perelman solved the Poincaré conjecture in 2003 (but refused the prize money in 2010).

https://en.wikipedia.org/wiki/Poincaré_conjecture

The Riemann hypothesis is one of the seven Millennium Prize Problems and is concerned with the zeros of the Riemann zeta function $\zeta(s)$.

Recall that $\zeta(1)$ is the harmonic series, which diverges. For complex values of $s \neq 1$, there is a unique way to "analytically continue" the function $\zeta(s)$ from the definition $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ which only works for $\operatorname{Re} s > 1$.

It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, ... These are called the trivial zeros of $\zeta(s)$.

The **Riemann hypothesis** claims that all other zeros of $\zeta(s)$ lie on the (vertical) line $s = \frac{1}{2} + ai$ (where a is real).

A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth $1,000,000.

http://www.claymath.org/millennium-problems/riemann-hypothesis

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The reason for caring about the zeros is that they are intimately tied to the distribution of primes. The prime number theorem states that, up to x, there are about $x / \ln(x)$ many primes. The Riemann hypothesis gives very precise error estimates for an improved prime number theorem (using a function more complicated than the logarithm).

The connection to primes. Here's a vague indication that $\zeta(s)$ is intimately connected to prime numbers:

$$\begin{aligned}\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s} &= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \ldots\right)\cdots \\ &= \frac{1}{1 - 2^{-s}} \frac{1}{1 - 3^{-s}} \frac{1}{1 - 5^{-s}} \cdots \\ &= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}\end{aligned}$$

To see that the second equality holds, imagine multiplying out the right-hand side. For instance, we get the term $\frac{1}{2^s} \cdot \frac{1}{3^{4s}} \cdot \frac{1}{7^s} = \frac{1}{(2 \cdot 3^4 \cdot 7)^s}$. This matches the term $\frac{1}{n^s}$ on the left-hand side for $n = 2 \cdot 3^4 \cdot 7$. Since every positive integer n has a unique factorization into prime factors, this matching is one-to-one.

The final infinite product is called the Euler product for the zeta function.

If the Riemann hypothesis was true, then we would be better able to estimate the number $\pi(x)$ of primes $p \leqslant x$. More generally, certain statements about the zeta function can be translated to statements about primes. For instance, the (non-obvious!) fact that $\zeta(s)$ has no zeros for $\operatorname{Re} s = 1$ implies the prime number theorem.

http://www-users.math.umn.edu/~garrett/m/v/pnt.pdf