

Review. Theorem 111: If x_0 is an ordinary point of a linear IVP, then it is guaranteed to have a power series solution $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.

Moreover, its radius of convergence is at least the distance between x_0 and the closest singular point.

Example 115. Find a minimum value for the radius of convergence of a power series solution to $(x^2 + 4)y'' - 3xy' + \frac{1}{x+1}y = 0$ at $x = 2$.

Solution. The singular points are $x = \pm 2i, -1$. Hence, $x = 2$ is an ordinary point of the DE and the distance to the nearest singular point is $|2 - 2i| = \sqrt{2^2 + 2^2} = \sqrt{8}$ (the distances are $|2 - (-1)| = 3, |2 - (\pm 2i)| = \sqrt{8}$). By Theorem 111, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{8}$.

Example 116. (caution!) Theorem 111 only holds for linear DEs! For nonlinear DEs, it is very hard to predict whether there is a power series solution and what its radius of convergence is.

Consider, for instance, the nonlinear DE $y' - y^2 = 0$.

Its coefficients have no singularities. A solution to this DE is $y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (see Example 120), which clearly has a problem at $x = 1$ (the radius of convergence is 1).

On the other hand. $y(x)$ also solves the linear DE $(1-x)y' - y = 0$ (or, even simpler, the order 0 "differential" equation $(1-x)y = 1$). Note how the DE has the singular point $x = 1$. Theorem 111 then allows us to predict that $y(x)$ must have a power series with radius of convergence at least 1.

Example 117. (Bessel functions) Consider the DE $x^2y'' + xy' + x^2y = 0$. Derive a recursive description of a power series solutions $y(x)$ at $x = 0$.

Caution! Note that $x = 0$ is a singular point (the only) of the DE. Theorem 111 therefore does not guarantee a basis of power series solutions. [However, $x = 0$ is what is called a **regular singular point**; for these, we are guaranteed one power series solution, as well as additional solutions expressed using logarithms and power series.]

Comment. We could divide the DE by x (but that wouldn't really change the computations below). The reason for not dividing that x is that this DE is the special case $\alpha = 0$ of the **Bessel equation** $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ (for which no such dividing is possible).

Solution. (plug in power series) Let us spell out power series for x^2y, xy', x^2y'' starting with $y(x) = \sum_{n=0}^{\infty} a_n x^n$:

$$x^2y(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$xy'(x) = \sum_{n=1}^{\infty} n a_n x^n \quad (\text{because } y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1})$$

$$x^2y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^n \quad (\text{because } y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2})$$

Hence, the DE becomes $\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$. We compare coefficients of x^n :

- $n = 1: a_1 = 0$
- $n \geq 2: n(n-1)a_n + n a_n + a_{n-2} = 0$, which simplifies to $n^2 a_n = -a_{n-2}$.

It follows that $a_{2n} = \frac{(-1)^n}{4^n n!^2} a_0$ and $a_{2n+1} = 0$.

Observation. The fact that we found $a_1 = 0$ reflects the fact that we cannot represent the general solution through power series alone.

Comment. If $a_0 = 1$, the function we found is a **Bessel function** and denoted as $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2}\right)^{2n}$.

The more general Bessel functions $J_\alpha(x)$ are solutions to the DE $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$.

Example 118. (caution!) Consider the linear DE $x^2y' = y - x$. Does it have a convergent power series solution at $x = 0$?

Important note. The DE $x^2y' = y - x$ has the singular point $x = 0$. Hence, Theorem 111 does not apply.

Advanced. Moreover, in contrast to the previous example, $x = 0$ is not a **regular singular point**. Indeed, as we see below, there is no power series solution of the DE at all.

Solution. Let us look for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$x^2y'(x) = x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n$$

Hence, $x^2y' = y - x$ becomes $\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n = \sum_{n=0}^{\infty} a_n x^n - x$. We compare coefficients of x^n :

- $n = 0$: $a_0 = 0$.
- $n = 1$: $0 = a_1 - 1$, so that $a_1 = 1$.
- $n \geq 2$: $(n-1)a_{n-1} = a_n$, from which it follows that $a_n = (n-1)a_{n-1} = (n-1)(n-2)a_{n-2} = \cdots = (n-1)!a_1 = (n-1)!$.

Hence the DE has the “formal” power series solution $y(x) = \sum_{n=1}^{\infty} (n-1)!x^n$.

However, that series is divergent for all $x \neq 0$; that is, the radius of convergence is 0.

Power series of familiar functions, continued

Example 119. The **hyperbolic cosine** $\cosh(x)$ is defined to be the even part of e^x . In other words, $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$. Determine its power series.

Solution. It follows from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

Comment. Note that $\cosh(ix) = \cos(x)$ (because $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$).

Comment. The hyperbolic sine $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ is similarly defined to be the odd part of e^x .

Example 120. (geometric series) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Why? If $y(x) = \sum_{n=0}^{\infty} x^n$, then $xy = y - 1$ (write down the power series for both sides!). Hence, $y = \frac{1}{1-x}$.

Alternatively, start with $y = \frac{1}{1-x}$ and note that y solves the order 0 “differential” (inhomogeneous) equation $(1-x)y = 1$. We can then determine a power series solution as we did in Example 105 to find $y = \sum_{n=0}^{\infty} x^n$.

Example 121. Determine a power series for $\frac{1}{1+x^2}$.

Solution. Replace x with $-x^2$ in $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (geometric series!) to get $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

Example 122. Determine a power series for $\arctan(x)$.

Solution. Recall that $\arctan(x) = \int \frac{dx}{1+x^2} + C$. Hence, we need to integrate $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

It follows that $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$. Since $\arctan(0) = 0$, we conclude that $C = 0$.

Example 123. Determine a power series for $\ln(x)$ around $x = 1$.

Solution. This is equivalent to finding a power series for $\ln(x+1)$ around $x = 0$ (see the final step).

Observe that $\ln(x+1) = \int \frac{dx}{1+x} + C$ and that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$.

Integrating, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$. Since $\ln(1) = 0$, we conclude that $C = 0$.

Finally, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is equivalent to $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$.

Comment. Choosing $x = 2$ in $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$ results in $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

The latter is the alternating harmonic sum.

Can you see from the series for $\ln(x)$ why the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges?

Example 124. (error function) Determine a power series for $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

Solution. It follows from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$.

Integrating, we obtain $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

Example 125. Determine the first several terms (up to x^5) in the power series of $\tan(x)$.

Solution. Observe that $y(x) = \tan(x)$ is the unique solution to the IVP $y' = 1 + y^2$, $y(0) = 0$.

We can therefore proceed to determine the first few power series coefficients as we did earlier.

That is, we plug $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$ into the DE. Note that $y(0) = 0$ means $a_0 = 0$.

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$1 + y^2 = 1 + (a_1x + a_2x^2 + a_3x^3 + \dots)^2 = 1 + a_1^2x^2 + (2a_1a_2)x^3 + (2a_1a_3 + a_2^2)x^4 + \dots$$

Comparing coefficients, we find: $a_1 = 1$, $2a_2 = 0$, $3a_3 = a_1^2$, $4a_4 = 2a_1a_2$, $5a_5 = 2a_1a_3 + a_2^2$.

Solving for a_2, a_3, \dots , we conclude that $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$

Comment. The fact that $\tan(x)$ is an odd function translates into $a_n = 0$ when n is even. If we had realized that at the beginning, our computation would have been simplified.

Advanced comment. The full power series is $\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1}$.

Here, the numbers B_{2n} are (rather mysterious) rational numbers known as **Bernoulli numbers**.

The radius of convergence is $\pi/2$. Note that this is not at all obvious from the DE $y' = 1 + y^2$. This illustrates the fact that nonlinear DEs are much more complicated than linear ones. (There is no analog of Theorem 111.)