Review. If $y(x)$ is "nice" at $x = x_0$ (i.e. analytic around $x = x_0$), then

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

In particular, at $x = 0$,

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \ldots$$

Power series solutions to linear DEs

Note how in the last two examples the "plug in power series" approach was complicated by the fact that the DE was not linear (we had to expand y^2 as well as $\cos(x+y)$, respectively).

For linear DEs, this complication does not arise and we can readily determine the complete power series expansion of analytic solutions (with a recursive description of the coefficients).

Example 105. (Airy equation, part II) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the power series of $y(x)$.

Solution. (plug in power series) Let us spell out the power series for y, y', y'' and xy:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Hence, $y'' = xy$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$. We compare coefficients of x^n:

- $n = 0$: $2 \cdot 1 a_2 = 0$, so that $a_2 = 0$.
- $n \geqslant 1$: $(n+2)(n+1) a_{n+2} = a_{n-1}$
  Replacing n by $n-2$, this is equivalent to $n(n-1) a_n = a_{n-3}$ for $n \geqslant 3$.

In conclusion, $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = a$, $a_1 = b$, $a_2 = 0$ as well as, for $n \geqslant 3$, $a_n = \frac{1}{n(n-1)} a_{n-3}$.

First few terms. In particular, $y = a\left(1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{(2 \cdot 3)(5 \cdot 6)} + \ldots\right) + b\left(x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{(3 \cdot 4)(6 \cdot 7)} + \ldots\right)$.

Advanced. The solution with $y(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$ and $y'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}$ is known as the **Airy function** $\mathrm{Ai}(x)$.
[A more natural property of $\mathrm{Ai}(x)$ is that it satisfies $y(x) \to 0$ as $x \to \infty$.]

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(Unless we specify otherwise, power series are meant to be about $x = 0$.)

Example 106. Determine the power series for $\cos(x)$ at $x = 0$.

Solution. Let $y(x) = \cos(x)$. After computing a few derivatives, we realize that $y^{(2n)}(x) = (-1)^n \cos(x)$ and $y^{(2n+1)}(x) = -(-1)^n \sin(x)$. In particular, $y^{(2n)}(0) = (-1)^n$ and $y^{(2n+1)}(0) = 0$. It follows that

$$\cos(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Comment. Note that the above observations on $y^{(2n)}$ and $y^{(2n+1)}$ simply reflect the fact that $\cos(x)$ is the unique solution to the IVP $y'' = -y$, $y(0) = 1$, $y'(0) = 0$.

Alternatively. We can also deduce the power series via Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$. Since

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(ix)^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!},$$

we conclude that $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

Example 107. Determine the first several terms in the power series of $\sin(2x^3)$ at $x = 0$.

Solution. (direct—unpleasant) If $f(x) = \sin(2x^3)$, then $f'(x) = 6x^2 \cos(2x^3)$ as well as $f''(x) = 12x\cos(2x^3) - 36x^4 \sin(2x^3)$ and $f'''(x) = 12\cos(2x^3) - 216x^3 \sin(2x^3) + 216x^6 \cos(2x^3)$.

In particular, $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$ and $f'''(0) = 12$.

It follows that $f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + \ldots = 0 + 0x + 0x^2 + \frac{12}{3!} x^3 + \ldots = 2x^3 + \ldots$

Solution. (via series for sine) As done in the previous example for $\cos(x)$, we can derive that

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \ldots$$

It follows that

$$\begin{aligned} \sin(2x^3) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x^3)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{6n+3} \\ &= \frac{2^1}{1!} x^3 - \frac{2^3}{3!} x^9 + \frac{2^5}{5!} x^{15} - \ldots = 2x^3 - \frac{4}{3} x^9 + \frac{4}{15} x^{15} - \ldots \end{aligned}$$

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