Some special functions and the power series method

Review: power series

Definition 98. A function $y(x)$ is analytic around $x = x_0$ if it has a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Note. In the next theorem, we will see that this power series is the Taylor series of $y(x)$ around $x = x_0$.

Power series are very pleasant to work with because they behave just like polynomials. For instance, we can differentiate and integrate them:

- If $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ (another power series!).

  We can rewrite the series as $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$.

  The result is a power series just like the one we started with. Likewise, for higher derivatives.

- $\int y(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C$

Theorem 99. If $y(x)$ is analytic around $x = x_0$, then $y(x)$ is infinitely differentiable and

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

Caution. Analyticity is needed in this theorem; being infinitely differentiable is not enough. For instance, $y(x) = e^{-1/x^2}$ is infinitely differentiable around $x = 0$ (and everywhere else). However, $y^{(n)}(0) = 0$ for all n.

In particular, if $y(x)$ is analytic at $x = 0$, then

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0) x + \frac{1}{2} y''(0) x^2 + \frac{1}{6} y'''(0) x^3 + \ldots$$

We have already seen the following example.

Example 100. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \ldots$

Once again, notice how the power series clearly has the property that $y' = y$ (as well as $y(0) = 1$).

It follows from here that, for instance, $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \ldots$

Example 101. Determine the power series for $7e^{3x}$ (at $x = 0$).

Solution. Instead of starting from scratch, we can use that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to conclude that

$$7e^{3x} = 7 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{7 \cdot 3^n}{n!} x^n = 7 + 21x + \frac{63}{2} x^2 + \frac{63}{2} x^3 + \frac{189}{8} x^4 + \ldots$$

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Power series solutions to DE

Given any DE, we can approximate analytic solutions by working with the first few terms of the power series.

Example 102. (Airy equation, part I) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the first several terms (up to x^6) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 \cdot y(0) = 0$.

Differentiating both sides of the DE, we obtain $y''' = y + xy'$ so that $y'''(0) = y(0) + 0 \cdot y'(0) = a$.

Likewise, $y^{(4)} = 2y' + xy''$ shows $y^{(4)}(0) = 2y'(0) = 2b$.

Continuing, $y^{(5)} = 3y'' + xy'''$ so that $y^{(5)}(0) = 3y''(0) = 0$.

Continuing, $y^{(6)} = 4y''' + xy^{(4)}$ so that $y^{(6)}(0) = 4y'''(0) = 4a$.

Hence, $y(x) = a + bx + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \frac{1}{120}y^{(5)}(0)x^5 + \frac{1}{720}y^{(6)}(0)x^6 + \ldots$
$= a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

Comment. Do you see the general pattern? We will revisit this example soon.

Solution. (plug in power series) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ becomes $y = a + bx + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial conditions.

To determine a_2, a_3, a_4, a_5, a_6, we equate the coefficients of:

$$\begin{aligned} y'' &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \ldots \\ xy &= ax + bx^2 + a_2x^3 + a_3x^4 + \ldots \end{aligned}$$

We find $2a_2 = 0$, $6a_3 = a$, $12a_4 = b$, $20a_5 = a_2$, $30a_6 = a_3$.

So $a_2 = 0$, $a_3 = \frac{a}{6}$, $a_4 = \frac{b}{12}$, $a_5 = \frac{a_2}{20} = 0$, $a_6 = \frac{a_3}{30} = \frac{a}{180}$. Hence, $y(x) = a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

Notation. When working with power series $\sum_{n=0}^{\infty} a_n x^n$, we sometimes write $O(x^n)$ to indicate that we omit terms that are multiples of x^n:

For instance. $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ or $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)$.

Example 103. Let $y(x)$ be the unique solution to the IVP $y' = x^2 + y^2$, $y(0) = 1$.
Determine the first several terms (up to x^4) in the power series of $y(x)$.

Solution. (successive differentiation—for humans) From the DE, $y'(0) = 0^2 + y(0)^2 = 1$.

Differentiating both sides of the DE, we obtain $y'' = 2x + 2yy'$. In particular, $y''(0) = 2$.

Continuing, $y''' = 2 + 2(y')^2 + 2yy''$ so that $y'''(0) = 2 + 2 + 2 \cdot 2 = 8$.

Likewise, $y^{(4)} = 6y'y'' + 2yy'''$ so that $y^{(4)}(0) = 12 + 16 = 28$.

Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \ldots = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

Comment. This approach requires the (symbolic) computation of intermediate derivatives. This is costly (even just the size of the simplified formulas is quickly increasing) and so the solution below is usually preferable for practical purposes. However, successive differentiation works well when working by hand.

Solution. (plug in power series—for computers) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ simplifies to $y = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial condition.

Therefore, $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$

To determine a_2, a_3, a_4, a_5, we need to expand $x^2 + y^2$ into a power series:

$y^2 = 1 + 2a_1x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_1a_2)x^3 + (2a_4 + 2a_1a_3 + a_2^2)x^4 + \ldots$ [we don't need the last term]

Equating coefficients of y' and $x^2 + y^2$, we find $a_1 = 1$, $2a_2 = 2a_1$, $3a_3 = 1 + 2a_2 + a_1^2$, $4a_4 = 2a_3 + 2a_1a_2$.

So $a_1 = 1$, $a_2 = 1$, $a_3 = \frac{4}{3}$, $a_4 = \frac{7}{6}$ and, hence, $y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

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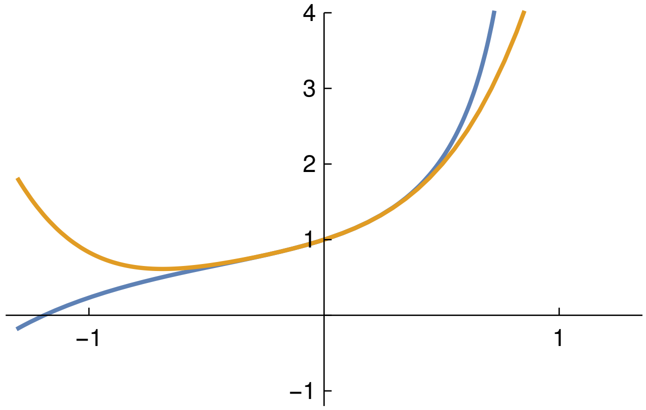
Below is a plot of $y(x)$ (in blue) and our approximation:



Note how the approximation is very good close to 0 but does not provide us with a "global picture".

Example 104. Let $y(x)$ be the unique solution to the IVP $y'' = \cos(x+y)$, $y(0)=0$, $y'(0)=1$. Determine the first several terms (up to x^5) in the power series of $y(x)$.

Solution. (successive differentiation—for humans) From the DE, $y''(0) = \cos(0 + y(0)) = 1$.
Differentiating both sides of the DE, we obtain $y''' = -\sin(x+y)(1+y')$.
In particular, $y'''(0) = -\sin(0+y(0))(1+y'(0)) = 0$.
Likewise, $y^{(4)} = -\cos(x+y)(1+y')^2 - \sin(x+y)y''$ shows $y^{(4)}(0) = -1\cdot 2^2 - 0 = -4$.
Continuing, $y^{(5)} = \sin(x+y)(1+y')^3 - 3\cos(x+y)(1+y')y'' - \sin(x+y)y'''$ so that $y^{(5)}(0) = -6$.
Hence, $y(x) = x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \frac{1}{120}y^{(5)}(0)x^5 + \ldots = x + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{20}x^5 + \ldots$

Solution. (plug in power series—for computers) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ simplifies to $y = x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial conditions.
Therefore, $y' = 1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$ and $y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots$
To determine a_2, a_3, a_4, a_5, we need to expand $\cos(x+y)$ into a power series:

Recall that $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$

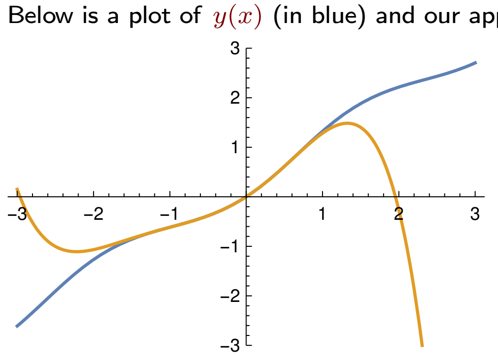
Hence, $\cos(x+y) = 1 - \frac{1}{2}(x+y)^2 + \frac{1}{24}(x+y)^4 + \ldots = 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2 + O(x^4)$.

Since $y^2 = (x + a_2x^2 + a_3x^3 + \ldots)^2 = x^2 + 2a_2x^3 + O(x^4)$,

$\cos(x+y) = 1 - \frac{1}{2}x^2 - x(x + a_2x^2) - \frac{1}{2}(x^2 + 2a_2x^3) + O(x^4) = 1 - 2x^2 - 2a_2x^3 + O(x^4)$.

Equating coefficients of y'' and $\cos(x+y)$, we find $2a_2 = 1$, $6a_3 = 0$, $12a_4 = -2$, $20a_5 = -2a_2$.
So $a_2 = \frac{1}{2}$, $a_3 = 0$, $a_4 = -\frac{1}{6}$, $a_5 = -\frac{1}{20}$ and, hence, $y(x) = x + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{20}x^5 + \ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



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