In the special case that $\Phi(t) = e^{At}$, some things become easier. For instance, $\Phi(t)^{-1} = e^{-At}$. In that case, we can explicitely write down solutions to IVPs:

- $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c}$.
- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{c} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$.

Example 96. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(a) Determine e^{At}.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution.

(a) By proceeding as in Example 67 (do it!), we find $e^{At} = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}$.

(b) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix}$

(c) $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$. We compute:

$\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \int_0^t \begin{bmatrix} 2e^{-2s} - e^{-3s} & -2e^{-2s} + 2e^{-3s} \\ e^{-2s} - e^{-3s} & -e^{-2s} + 2e^{-3s} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^s \end{bmatrix}\mathrm{d}s = \int_0^t \begin{bmatrix} -4e^{-s} + 4e^{-2s} \\ -2e^{-s} + 4e^{-2s} \end{bmatrix}\mathrm{d}s = \begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix}$

Hence, $e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}\begin{bmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix}$.

Finally, $\boldsymbol{y}(t) = \begin{bmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{bmatrix} + \begin{bmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix} = \begin{bmatrix} 2e^t - 6e^{2t} + 5e^{3t} \\ -3e^{2t} + 5e^{3t} \end{bmatrix}$.

Sage. Here is how we can let Sage do these computations for us:

```
>>> s, t = var('s, t')
>>> A = matrix([[1,2],[-1,4]])
>>> y = exp(A*t)*vector([1,2]) + exp(A*t)*integrate(exp(-A*s)*vector([0,2*e^s]), s,0,t)
>>> y.simplify_full()
```

$(5\ e^{(3\ t)} - 6\ e^{(2\ t)} + 2\ e^t,\ 5\ e^{(3\ t)} - 3\ e^{(2\ t)})$

Comment. Can you see that the solution is of the form that we anticipate from the method of undetermined coefficients?
Indeed, $\boldsymbol{y}(t) = \boldsymbol{y}_p(t) + \boldsymbol{y}_h(t)$ where the simplest particular solution is $\boldsymbol{y}_p(t) = \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}$.

Example 97. In Example 89, we derived the IVP $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.
Solve it using our new tools.

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Solution. This is an IVP that we can solve (with some work). Do it! For instance, we can apply variation of constants. (Alternatively, leverage our particular solution from the previous part!) Skipping most work, we find:

- If $A = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$, then $e^{At} = \frac{1}{7}\begin{bmatrix} e^{-9t}+6e^{-2t} & -2e^{-9t}+2e^{-2t} \\ -3e^{-9t}+3e^{-2t} & 6e^{-9t}+e^{-2t} \end{bmatrix}$
- $\boldsymbol{y} = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} 27 \\ 0 \end{bmatrix}\mathrm{d}s = \frac{1}{7}\begin{bmatrix} e^{-9t}+6e^{-2t} \\ -3e^{-9t}+3e^{-2t} \end{bmatrix} + \frac{3}{14}e^{At}\begin{bmatrix} 2e^{9t}+54e^{2t}-56 \\ -6e^{9t}+27e^{2t}-21 \end{bmatrix}$
  $= \begin{bmatrix} 12-9e^{-2t} \\ 4.5-4.5e^{-2t} \end{bmatrix}$

Application of variation of constants: the second order case

Review. In Theorem 94 we showed that $\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t)$ has the particular solution

$$\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t,$$

where $\Phi(t)$ is a fundamental matrix solution to $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$.

Let us apply this result to the case of a second-order LDE

$$y'' + P(t)y' + Q(t)y = F(t).$$

We can write this DE as a first-order system by introducing the vector $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -Q(t) & -P(t) \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

If the second-order homogeneous DE (that is, $y'' + P(t)y' + Q(t)y = 0$) has general solution $C_1y_1(t) + C_2y_2(t)$, then a fundamentral matrix for the first-order homogeneous system is

$$\Phi(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

(recall that each column of $\Phi(t)$ represents a solution $\boldsymbol{y}$ of the system). Our formula from Theorem 94 now gives us a particular solution of the inhomogeneous system:

$$\begin{aligned}
\boldsymbol{y}_p(t) &= \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t \\
&= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\int \frac{1}{y_1y_2' - y_1'y_2}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ F \end{bmatrix}\mathrm{d}t \\
&= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\int \frac{F}{y_1y_2' - y_1'y_2}\begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}\mathrm{d}t \\
&= \int \frac{-y_2F}{y_1y_2' - y_1'y_2}\mathrm{d}t\begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + \int \frac{y_1F}{y_1y_2' - y_1'y_2}\mathrm{d}t\begin{bmatrix} y_2 \\ y_2' \end{bmatrix}
\end{aligned}$$

The first entry of $\boldsymbol{y}_p$ is the corresponding particular solution to the second-order inhomogeneous DE:

$$y_p(t) = C_1(t)y_1(t) + C_2(t)y_2(t), \quad C_1(t) = \int \frac{-y_2(t)F(t)}{W(t)}\mathrm{d}t, \quad C_2(t) = \int \frac{y_1(t)F(t)}{W(t)}\mathrm{d}t.$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the **Wronskian**.