Review: Linear first-order DEs

The most general first-order linear DE is $P(t)y' + Q(t)y + R(t) = 0$.

By dividing by $P(t)$ and rearranging, we can always write it in the form $y' = a(t)y + f(t)$.

The corresponding **homogeneous** linear DE is $y' = a(t)y$.

Its general solution is $y(t) = Ce^{\int a(t)\mathrm{d}t}$.

Why? Compute y' and verify that the DE is indeed satisfied. Alternatively, we can derive the formula using separation of variables as illustrated in the next example.

Example 91. (review homework) Solve $y' = t^2 y$.

Solution. This DE is separable as well: $\frac{1}{y}\mathrm{d}y = t^2\,\mathrm{d}t$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = \frac{1}{3}t^3 + A$, so that $|y| = e^{\frac{1}{3}t^2 + A}$. Since the RHS is never zero, we must have either $y = e^{\frac{1}{3}t^2 + A}$ or $y = -e^{\frac{1}{3}t^2 + A}$.

Hence $y = \pm e^A e^{\frac{1}{3}t^3} = Ce^{\frac{1}{3}t^3}$ (with $C = \pm e^A$). Note that $C = 0$ corresponds to the singular solution $y = 0$.

In summary, the general solution is $y = Ce^{\frac{1}{3}t^3}$ (with C any real number).

Solving linear first-order DEs using variation of constants

Recall that, to find the general solution of the **inhomogeneous DE**

$$y' = a(t)y + f(t),$$

we only need to find a particular solution y_p.

Then the general solution is $y_p + Cy_h$, where y_h is any solution of the homogeneous DE $y' = a(t)y$.

Comment. In applications, $f(t)$ often represents an external force. As such, the inhomogeneous DE is sometimes called "driven" while the homogeneous DE would be called "undriven".

Theorem 92. (variation of constants) $y' = a(t)y + f(t)$ has the particular solution

$$y_p(t) = c(t)y_h(t) \quad \text{with } c(t) = \int \frac{f(t)}{y_h(t)}\mathrm{d}t,$$

where $y_h(t) = e^{\int a(t)\mathrm{d}t}$ is a solution to the homogeneous equation $y' = a(t)y$.

Proof. Let us plug $y_p(t) = y_h(t)\int \frac{f(t)}{y_h(t)}\mathrm{d}t$ into the DE to verify that it is a solution:

$$y_p'(t) = y_h'(t)\int \frac{f(t)}{y_h(t)}\mathrm{d}t + y_h(t)\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{f(t)}{y_h(t)}\mathrm{d}t}_{\frac{f(t)}{y_h(t)}} = a(t)y_h(t)\int \frac{f(t)}{y_h(t)}\mathrm{d}t + f(t) = a(t)y_p(t) + f(t) \qquad \square$$

Comment. Note that the formula for $y_p(t)$ gives the general solution if we let $\int \frac{f(t)}{y_h(t)}\mathrm{d}x$ be the general antiderivative. (Think about the effect of the constant of integration!)

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Example 93. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. To apply Theorem 92, we write as $\frac{dy}{dx} = a(x)y + f(x)$ with $a(x) = -\frac{1}{x}$ and $f(x) = \frac{1}{x^2} + \frac{2}{x}$.
$y_h(x) = e^{\int a(x)dx} = e^{-\ln x} = \frac{1}{x}$. (Why can we write $\ln x$ instead of $\ln|x|$? See comment below.) Hence:

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)} dx = \frac{1}{x} \int \left(\frac{1}{x} + 2\right) dx = \frac{\ln x + 2x + C}{x}$$

Using $y(1) = 3$, we find $C = 1$. In summary, the solution is $y = \frac{\ln(x) + 2x + 1}{x}$.

Comment. Note that $x = 1 > 0$ in the initial condition. Because of that we know that (at least locally) our solution will have $x > 0$. Accordingly, we can use $\ln x$ instead of $\ln|x|$. (If the initial condition had been $y(-1) = 3$, then we would have $x < 0$, in which case we can use $\ln(-x)$ instead of $\ln|x|$.)

Comment. Observe how the general solution (with parameter C) is indeed obtained from any particular solution (say, $\frac{\ln x + 2x}{x}$) plus the general solution to the homogeneous equation, which is $\frac{C}{x}$.

How to find the formula for variation of constants?

- **Variation of constants** means that we look for a solution of the form $y_p(t) = c(t) y_h(t)$.

  Keep in mind that $c y_h(t)$ is the solution to the homogeneous DE. Going from a constant c (for the homogeneous case) to $c(t)$ (for the inhomogeneous case) is why this is called "**variation of constants**" (or, sometimes, variation of parameters).

- To find a $c(t)$ that works, we plug into the DE $y' = ay + f$ which results in

  $$c' y_h + c y_h' = a c y_h + f.$$

  Since $y_h' = a y_h$, this simplifies to $c' y_h = f$ or, equivalently, $c' = \frac{f}{y_h}$.

- We integrate to find $c(t) = \int \frac{f(t)}{y_h(t)} dt$, which is the formula in Theorem 92.

How does this compare to the integrating factor method? Instead of variation of constants, you may have solved linear DEs using **integrating factors** instead. In that case, the DE is first written as $y' - a(t)y = f(t)$ and then both sides are multiplied with the integrating factor

$$g(t) = \exp\left(\int -a(t) dt\right).$$

Because $g'(t) = -a(t)g(t)$, we then have

$$\underbrace{g(t)y' - a(t)g(t)y}_{= \frac{d}{dt} g(t) y} = f(t)g(t).$$

Integrating both sides gives

$$g(t) y = \int f(t) g(t) dt.$$

Since $g(t) = 1 / y_h(t)$, this then produces the same formula for y that we found using variation of constants.

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Systems of linear DEs: the inhomogeneous case

Recall that any linear DE can be transformed into a first-order system. Hence, any linear DE (or any system of linear DEs) can be written as

$$\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t).$$

Note. The DE is allowed to have nonconstant coefficients (A depends on t). On the other hand, this is an autonomous DE (those for which we can analyze phase portraits) only if $A(t)$ and $\boldsymbol{f}(t)$ actually don't depend on t.

The same arguments as for Theorem 92 with the same result apply to systems of linear equations!

Recall that we showed in Theorem 92 that $y' = a(t)y + f(t)$ has the particular solution

$$y_p(t) = y_h(t)\int \frac{f(t)}{y_h(t)}\mathrm{d}t,$$

where $y_h(t) = e^{\int a(t)\mathrm{d}t}$ is a solution to the homogeneous equation $y' = a(t)y$.

Theorem 94. (variation of constants) $\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t)$ has the particular solution

$$\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t,$$

where $\Phi(t)$ is a fundamental matrix solution to $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$.

Proof. Since the general solution of the homogeneous equation $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$ is $\boldsymbol{y}_h = \Phi(t)\boldsymbol{c}$, we are going to vary the constant $\boldsymbol{c}$ and look for a particular solution of the form $\boldsymbol{y}_p = \Phi(t)\boldsymbol{c}(t)$. Plugging into the DE, we get:

$\boldsymbol{y}_p' = \Phi'\boldsymbol{c} + \Phi\boldsymbol{c}' = A\Phi\boldsymbol{c} + \Phi\boldsymbol{c}' \stackrel{!}{=} A\boldsymbol{y}_p + \boldsymbol{f} = A\Phi\boldsymbol{c} + \boldsymbol{f}$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used $\Phi' = A\Phi$.

Hence, $\boldsymbol{y}_p = \Phi(t)\boldsymbol{c}(t)$ is a particular solution if and only if $\Phi\boldsymbol{c}' = \boldsymbol{f}$.

The latter condition means $\boldsymbol{c}' = \Phi^{-1}\boldsymbol{f}$ so that $\boldsymbol{c} = \int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t$, which gives the claimed formula for $\boldsymbol{y}_p$. □

Example 95. Find a particular solution to $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ -2e^{3t} \end{bmatrix}$.

Solution. First, we determine (do it!) a fundamental matrix solution for $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$: $\Phi(x) = \begin{bmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{bmatrix}$

Using $\det(\Phi(t)) = 5e^{3t}$, we find $\Phi(t)^{-1} = \frac{1}{5}\begin{bmatrix} 2e^{t} & -3e^{t} \\ e^{-4t} & e^{-4t} \end{bmatrix}$.

Hence, $\Phi(t)^{-1}\boldsymbol{f}(t) = \frac{2}{5}\begin{bmatrix} 3e^{4t} \\ -e^{-t} \end{bmatrix}$ and $\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}x = \frac{2}{5}\begin{bmatrix} 3/4e^{4t} \\ e^{-t} \end{bmatrix}$.

By variation of constants, $\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t = \begin{bmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{bmatrix}\frac{2}{5}\begin{bmatrix} 3/4e^{4t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}e^{3t}$.

Comment. Note that the solution is of the form that we anticipate from the method of undetermined coefficients (which we only discussed in the case of a single DE but which works similarly for systems).

Sage. Here is a way to have Sage do these computations for us. Keep in mind that we can choose $\Phi(t) = e^{At}$.

```
>>> s, t = var('s, t')
>>> A = matrix([[2,3],[2,1]])
>>> y = exp(A*t)*integrate(exp(-A*t)*vector([0,-2*e^(3*t)]), t)
>>> y.simplify_full()
```

$$\left(\frac{3}{2}\,e^{(3\,t)}, \frac{1}{2}\,e^{(3\,t)}\right)$$