Stability of autonomous linear differential equations

Example 82. (spiral source, spiral sink, center point)

(a) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

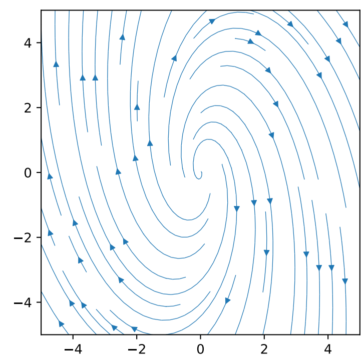
(c) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Solution.

(a)

The eigenvalues are $\lambda = 1 \pm 2i$ and the general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}e^t + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}e^t$$



In this case, the origin is a **spiral source** which is an unstable equilibrium (note that it follows from $e^t \to \infty$ as $t \to \infty$ that all solutions "flow away" from the origin because they have increasing amplitude).

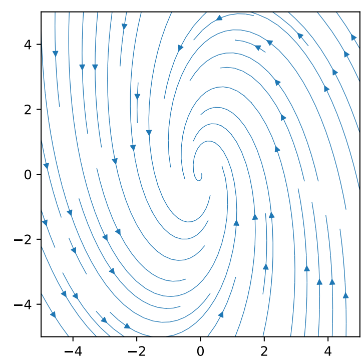
Review. $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ parametrizes the unit circle.

Similarly, $\begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$ parametrizes an ellipse.

(b)

The eigenvalues are $\lambda = -1 \pm 2i$ and the general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}e^{-t} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}e^{-t}$$



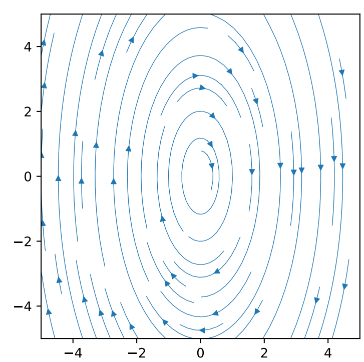
In this case, the origin is a **spiral sink** which is an asymptotically stable equilibrium (note that it follows from $e^{-t} \to 0$ as $t \to \infty$ that all solutions "flow into" the origin because their amplitude goes to zero).

Comment. Note that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ solves the first system if and only if $\begin{bmatrix} x(-t) \\ y(-t) \end{bmatrix}$ is a solution to the second. Consequently, the phase portraits look alike but all arrows are reversed.

(c)

The eigenvalues are $\lambda = \pm 2i$ and the general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$



In this case, the origin is a **center point** which is a stable equilibrium (note that the solutions are periodic with period π and therefore loop around the origin; with each trajectory a perfect ellipse).

Armin Straub
straub@southalabama.edu

Review. In Example 77, we considered the system $\frac{dx}{dt} = y - 5x$, $\frac{dy}{dt} = 4x - 2y$.

We found that it has general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

In particular, the only equilibrium point is $(0,0)$ and it is asymptotically stable.

The following example is an inhomogeneous version of Example 77:

Example 83. Analyze the system $\frac{dx}{dt} = y - 5x + 3$, $\frac{dy}{dt} = 4x - 2y$.

In particular, determine the general solution as well as all equilibrium points and their stability.

Solution. As reviewed above, we looked at the corresponding homogeneous system in Example 77 and found that its general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

Note that we can write the present system in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$.

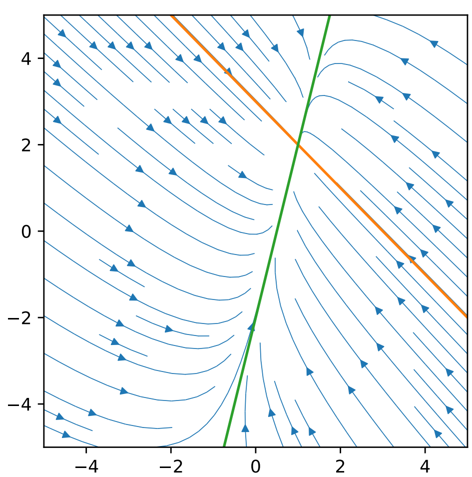
To find the equilibrium point, we solve $M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 0$ to find $\begin{bmatrix} x \\ y \end{bmatrix} = -M^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The fact that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an equilibrium point means that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a particular solution!

(Make sure that you see that it has exactly the form we expect from the method of undetermined coefficients!)

Thus, the general solution must be $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$ (that is, the particular solution plus the general solution of the homogeneous system that we solved in Example 77).

As a result, the phase portrait is going to look just as in Example 77 but shifted by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$:



Because both eigenvalues (-1 and -6) are negative, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an asymptotically stable equilibrium point. More precisely, it is what is called a **nodal source**.

Armin Straub
straub@southalabama.edu

As we have started to observe, the eigenvalues determine the stability of the equilibrium point in the case of an autonomous linear 2-dimensional systems. The following table gives an overview.

Important. Note that such a system must be of the form $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. Because the system is autonomous, the matrix M and the inhomogeneous part $\boldsymbol{c}$ cannot depend on t.

(stability of autonomous linear 2-dimensional systems)

eigenvalues	behaviour	stability	solutions have terms like
real and both positive	nodal source	unstable	e^{3t}, e^{7t}
real and both negative	nodal sink	asymptotically stable	e^{-3t}, e^{-7t}
real and opposite signs	saddle	unstable	e^{-3t}, e^{7t}
complex with positive real part	spiral source	unstable	$e^{3t}\cos(7t)$, $e^{3t}\sin(7t)$
complex with negative real part	spiral sink	asymptotically stable	$e^{-3t}\cos(7t)$, $e^{-3t}\sin(7t)$
purely imaginary	center point	stable (not asymptotically stable)	$\cos(7t)$, $\sin(7t)$

Stability of nonlinear autonomous systems

We now observe that we can (typically) determine the stability of an equilibrium point of a nonlinear system by simply linearizing at that point.

(stability of autonomous nonlinear 2-dimensional systems)

Suppose that (x_0, y_0) is an equilibrium point of the system $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$.

If the Jacobian matrix $J(x_0, y_0)$ is invertible, then its eigenvalues determine the stability and behaviour of the equlibrium point as for a linear system except in the following cases:

- If the eigenvalues are pure imaginary, we cannot predict stability (the equilibrium point could be either a center or a spiral source/sink; whereas the equilibrium point of the linearization is a center).
- If the eigenvalues are real and equal, then the equilibrium point could be either nodal or spiral (whereas the linearization has a nodal equilibrium point). The stability, however, is the same.

Comment. We need the Jacobian matrix $J(x_0, y_0)$ to be invertible, so that the linearized system has a unique equilibrium point.

Plot, for instance, the phase portrait of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x-2y)x \\ (x-2)y \end{bmatrix}$.

Purely imaginary eigenvalues? The issue with pure imaginary eigenvalues here comes from the fact that the linearization is only an approximation, with the true (nonlinear) behaviour slightly deviating. Slightly perturbing purely imaginary roots can also lead to (small but) positive real part (unstable; spiral source) or negative real part (asymptotically stable; spiral sink).

Real repeated eigenvalue? The issue with a real repeated eigenvalue is similar. Slightly perturbing such a root can lead to real eigenvalues (nodal) or a pair of complex conjugate eigenvalues (spiral). However, the real part of these perturbations still has the same sign so that we can still predict the stability itself.

The following is a continuation of Example 75:

Example 84. (cont'd) Consider again the system $\frac{dx}{dt} = x \cdot (y-1)$, $\frac{dy}{dt} = y \cdot (x-1)$. Without consulting a plot, determine the equilibrium points and classify their stability.

Solution. See Example 75 for the phase portrait. However, we will not use it in the following.

To find the equilibrium points, we solve $x(y-1)=0$ (that is, $x=0$ or $y=1$) and $y(x-1)=0$ (that is, $x=1$ or $y=0$). We conclude that the equilibrium points are $(0,0)$ and $(1,1)$.

Our system is $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ with $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x \cdot (y-1) \\ y \cdot (x-1) \end{bmatrix}$.

The Jacobian matrix is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y-1 & x \\ y & x-1 \end{bmatrix}$.

- At $(0,0)$, the Jacobian matrix is $J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We can read off that the eigenvalues are $-1, -1$. Since they are both negative, $(0,0)$ is asymptotically stable.

  Since this is a real repeated eigenvalue, we cannot immediately tell whether $(0,0)$ is a nodal sink (it is a nodal sink for the linearization!) or a nodal spiral. (Since our system is nonlinear, the linearization is just an approximation. Similarly, you can think of the eigenvalues $-1, -1$ as being somewhat approximate. Slight jiggling could change them to something like $-1 \pm 0.001i$ which would correspond to a nodal spiral.)

- At $(1,1)$, the Jacobian matrix is $J(1,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

  The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1$, which has roots ± 1. These are the eigenvalues. Since one is positive and the other is negative, $(1,1)$ is a saddle. In particular, $(1,1)$ is unstable.

Example 85. Consider again the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y - x^2 \\ (x-3)(x-y) \end{bmatrix}$. Without consulting a plot, determine the equilibrium points and classify their stability.

Solution. To find the equilibrium points, we solve $2y - x^2 = 0$ and $(x-3)(x-y)$.
It follows from the second equation that $x = 3$ or $x = y$:

- If $x = 3$, then the first equation implies $y = \frac{9}{2}$.
- If $x = y$, then the first equation becomes $2y - y^2 = 0$, which has solutions $y = 0$ and $y = 2$.

Hence, the equilibrium points are $(0,0)$, $(2,2)$ and $\left(3, \frac{9}{2}\right)$.

The Jacobian matrix of $\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 2y - x^2 \\ (x-3)(x-y) \end{bmatrix}$ is $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -2x & 2 \\ 2x - y - 3 & -x + 3 \end{bmatrix}$.

- At $(0,0)$, the Jacobian matrix is $J = \begin{bmatrix} 0 & 2 \\ -3 & 3 \end{bmatrix}$. The eigenvalues are $\frac{1}{2}(3 \pm i\sqrt{15})$.

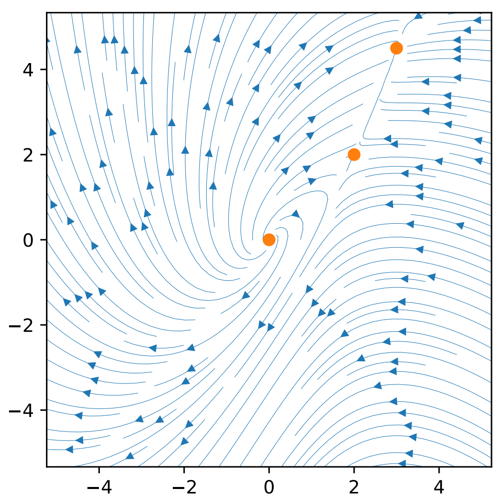
  Since these are complex with positive real part, $(0,0)$ is a spiral source and, in particular, unstable.

- At $(2,2)$, the Jacobian matrix is $J = \begin{bmatrix} -4 & 2 \\ -1 & 1 \end{bmatrix}$. The eigenvalues are $\frac{1}{2}(-3 \pm \sqrt{17}) \approx -3.562, 0.562$.

  Since these are real with opposite signs, $(2,2)$ is a saddle and, in particular, unstable.

- At $\left(3, \frac{9}{2}\right)$, the Jacobian matrix is $J = \begin{bmatrix} -6 & 2 \\ -\frac{3}{2} & 0 \end{bmatrix}$. The eigenvalues are $-3 \pm \sqrt{6} \approx -5.449, -0.551$.

  Since these are real and both negative, $\left(3, \frac{9}{2}\right)$ is a nodal sink and, in particular, asymptotically stable.



Comment. Can you confirm our analysis in the above plot? Note that it is becoming hard to see the details. One solution would be to make separate phase portraits focusing on the vicinity of each equilibrium plot. Do it!

Armin Straub
straub@southalabama.edu