Example 78. Consider the system $\frac{dx}{dt} = 5x - y$, $\frac{dy}{dt} = 2y - 4x$.

(a) Determine the general solution.

(b) Make a phase portrait.

(c) Determine all equilibrium points and their stability.

Solution.

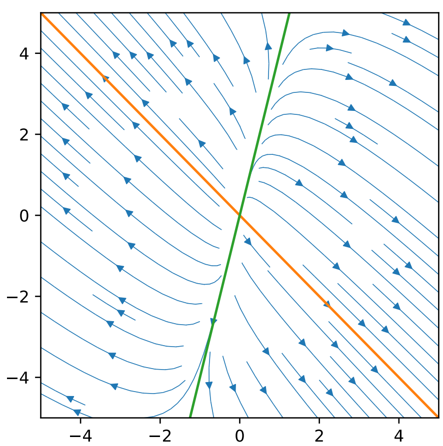
(a) Note that we can write this is in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = -\begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$, where the matrix is exactly -1 times what it was in Example 77.

Consequently, M has 1-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as 6-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (Can you explain why the eigenvectors are the same and the eigenvalues changed sign?)

Thus, the general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$.

(b) We again have Sage make the plot for us:

```
>>> x,y = var('x y')
    streamline_plot((5*x-y,-4*x+2*y), (x,-4,4), (y,-4,4))
```



Note that the phase portrait is identical to the one in Example 77, except that the arrows are reversed.

(c) The only equilibrium point is $(0,0)$ and it is unstable.

We can see this from the phase portrait but we can also see it readily from our general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ because e^t and e^{6t} go to ∞ as $t \to \infty$.

In general. If at least one eigenvalue is positive, then the equilibrium is unstable.

Example 79. Suppose the system $\frac{dx}{dt} = f(x, y)$, $\frac{dy}{dt} = g(x, y)$ has general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$. Determine all equilibrium points and their stability.

Solution. Recall that equilibrium points correspond to constant solutions. Clearly, the only constant solution is the zero solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Equivalently, the only equilibrium point is $(0,0)$.

Since $e^{6t} \to \infty$ as $t \to \infty$, we conclude that the equilibrium is unstable. (Note that the solution $C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ starts arbitrarily near to $(0,0)$ but always "flows away").

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Review: Linearizations of nonlinear functions

Recall from Calculus I that a function $f(x)$ around a point x_0 has the linearization

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Here, the right-hand side is the linearization and we also know it as the tangent line to $f(x)$ at x_0.

Recall from Calculus III that a function $f(x, y)$ around a point (x_0, y_0) has the linearization

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Again, the right-hand side is the linearization. This time, it describes the tangent plane to $f(x, y)$ at (x_0, y_0). Recall that $f_x = \frac{\partial}{\partial x} f(x, y)$ and $f_y = \frac{\partial}{\partial y} f(x, y)$ are the partial derivatives of f.

Example 80. Determine the linearization of the function $3 + 2xy^2$ at $(2, 1)$.

Solution. If $f(x, y) = 3 + 2xy^2$, then $f_x = 2y^2$ and $f_y = 4xy$. In particular, $f_x(2, 1) = 2$ and $f_y(2, 1) = 8$. Accordingly, the linearization is $f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 7 + 2(x - 2) + 8(y - 1)$.

It follows that a vector function $\boldsymbol{f}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$ around a point (x_0, y_0) has the linearization

$$\begin{aligned} \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} &\approx \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \begin{bmatrix} f_x(x_0, y_0) \\ g_x(x_0, y_0) \end{bmatrix}(x - x_0) + \begin{bmatrix} f_y(x_0, y_0) \\ g_y(x_0, y_0) \end{bmatrix}(y - y_0) \\ &= \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \underbrace{\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}}_{= J(x_0, y_0)} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}. \end{aligned}$$

The matrix $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$ is called the **Jacobian matrix** of $\boldsymbol{f}(x, y)$.

Example 81. Determine the linearization of the vector function $\begin{bmatrix} 3 + 2xy^2 \\ x(y^3 - 2x) \end{bmatrix}$ at $(2, 1)$.

Solution. If $\begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} = \begin{bmatrix} 3 + 2xy^2 \\ x(y^3 - 2x) \end{bmatrix}$, then the Jacobian matrix is

$$J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2y^2 & 4xy \\ y^3 - 4x & 3xy^2 \end{bmatrix}.$$

In particular, $J(2, 1) = \begin{bmatrix} 2 & 8 \\ -7 & 6 \end{bmatrix}$. The linearization is $\begin{bmatrix} f(2, 1) \\ g(2, 1) \end{bmatrix} + J(2, 1) \begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix}$.

Important comment. If we multiply out the matrix-vector product, then we get $\begin{bmatrix} 7 + 2(x - 2) + 8(y - 1) \\ -6 - 7(x - 2) + 6(y - 1) \end{bmatrix}$.

In the first component we get exactly what we got for the linearization of $f(x, y)$ in the previous example. Likewise, the second component is the linearization of $g(x, y)$ by itself.

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