

Example 59. (review) Write the (second-order) RE $a_{n+2} = a_{n+1} + 2a_n$, with $a_0 = 0$, $a_1 = 1$, as a system of (first-order) recurrences.

Solution. If $\mathbf{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, then $\mathbf{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+1} + 2a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{a}_n$ with $\mathbf{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 60. Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

- (a) Determine the general solution to $\mathbf{a}_{n+1} = M\mathbf{a}_n$.
- (b) Determine a fundamental matrix solution to $\mathbf{a}_{n+1} = M\mathbf{a}_n$.
- (c) Compute M^n .
- (d) Solve $\mathbf{a}_{n+1} = M\mathbf{a}_n$, $\mathbf{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

- (a) Recall that each λ -eigenvector \mathbf{v} of M provides us with a solution: namely, $\mathbf{a}_n = \mathbf{v}\lambda^n$.

The characteristic polynomial is: $\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 2 & 1 - \lambda \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. Hence, the eigenvalues are $\lambda = 2$ and $\lambda = -1$.

- $\lambda = 2$: Solving $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \mathbf{v} = \mathbf{0}$, we find that $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.
- $\lambda = -1$: Solving $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \mathbf{v} = \mathbf{0}$, we find that $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

Hence, the general solution is $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2^n + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (-1)^n$.

- (b) Note that $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2^n + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (-1)^n = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Hence, a fundamental matrix solution is $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix}$.

Comment. Other choices are possible and natural. For instance, the order of the two columns is based on our choice of starting with $\lambda = 2$. Also, the columns can be scaled by any constant (for instance, using $-\mathbf{v}$ instead of \mathbf{v} for $\lambda = -1$ above, we end up with the same Φ_n but with the second column scaled by -1).

In general, if Φ_n is a fundamental matrix solution, then so is $\Phi_n \mathbf{C}$ where \mathbf{C} is an invertible 2×2 matrix.

- (c) We compute $M^n = \Phi_n \Phi_0^{-1}$ using $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix}$. Since $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, we have

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2 \cdot 2^n - 2(-1)^n & 2 \cdot 2^n + (-1)^n \end{bmatrix}.$$

- (d) $\mathbf{a}_n = M^n \mathbf{a}_0 = \frac{1}{3} \begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2 \cdot 2^n - 2(-1)^n & 2 \cdot 2^n + (-1)^n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^n - (-1)^n \\ 2 \cdot 2^n + (-1)^n \end{bmatrix}$

Alternative solution of the first part. We saw in Example 59 that this system can be obtained from $a_{n+2} = a_{n+1} + 2a_n$ if we set $\mathbf{a} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. In Example 44, we found that this RE has solutions $a_n = 2^n$ and $a_n = (-1)^n$.

Correspondingly, $\mathbf{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{a}_n$ has solutions $\mathbf{a}_n = \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix}$ and $\mathbf{a}_n = \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$.

These combine to the general solution $C_1 \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix} + C_2 \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$ (equivalent to our solution above).

Alternative for last part. Solve the RE from Example 59 to find $a_n = \frac{1}{3}(2^n - (-1)^n)$. The above is $\mathbf{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$.

We have learned how to compute M^n for a matrix M using its eigenvalues and eigenvectors, as well as solve the system $\mathbf{a}_{n+1} = M\mathbf{a}_n$. For diagonal matrices, all this is much simpler:

Example 61. If $M = \begin{bmatrix} 3 & & & \\ & -2 & & \\ & & 5 & \\ & & & 1 \end{bmatrix}$, what is M^n ?

Also: what is the solution to $\mathbf{a}_{n+1} = M\mathbf{a}_n$?

Comment. Entries that are not printed are meant to be zero (to make the structure of the 4×4 matrix more visibly transparent).

Solution. $M^n = \begin{bmatrix} 3^n & & & \\ & (-2)^n & & \\ & & 5^n & \\ & & & 1 \end{bmatrix}$

If this isn't clear to you, multiply out M^2 . What happens?

Also: $\mathbf{a}_{n+1} = M\mathbf{a}_n$ with $\mathbf{a}_n = \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix}$ decouples into $\begin{aligned} a_{n+1} &= 3a_n \\ b_{n+1} &= -2b_n \\ c_{n+1} &= 5c_n \\ d_{n+1} &= d_n \end{aligned}$ which is solved by $\mathbf{a}_n = \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} = \begin{bmatrix} 3^n a_0 \\ (-2)^n b_0 \\ 5^n c_0 \\ d_0 \end{bmatrix}$.

Example 62. (extra practice)

- Write the recurrence $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ as a system $\mathbf{a}_{n+1} = M\mathbf{a}_n$ of (first-order) recurrences.
- Determine a fundamental matrix solution to $\mathbf{a}_{n+1} = M\mathbf{a}_n$.
- Compute M^n .

Solution.

(a) If $\mathbf{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$, then the RE becomes $\mathbf{a}_{n+1} = M\mathbf{a}_n$ with $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$.

(b) Because we started with a single (third-order) equation, we can avoid computing eigenvectors and eigenvalues (indeed, we will find these as a byproduct).

By factoring the characteristic equation $N^3 - 4N^2 + N + 6 = (N - 3)(N - 2)(N + 1)$, we find that the characteristic roots are $3, 2, -1$ (these are also precisely the eigenvalues of M).

Hence, $\mathbf{a}_n = C_1 \cdot 3^n + C_2 \cdot 2^n + C_3 \cdot (-1)^n$ is the general solution to the initial RE.

Correspondingly, a fundamental matrix solution of the system is $\Phi_n = \begin{bmatrix} 3^n & 2^n & (-1)^n \\ 3 \cdot 3^n & 2 \cdot 2^n & -(-1)^n \\ 9 \cdot 3^n & 4 \cdot 2^n & (-1)^n \end{bmatrix}$.

Note. This tells us that $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ is a 3-eigenvector, $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ a 2-eigenvector, and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ a -1 -eigenvector of M .

(c) Since $\Phi_{n+1} = M\Phi_n$, we have $\Phi_n = M^n\Phi_0$ so that $M^n = \Phi_n\Phi_0^{-1}$. This allows us to compute that:

$$M^n = \frac{1}{12} \begin{bmatrix} -6 \cdot 3^n + 12 \cdot 2^n + 6(-1)^n & -3 \cdot 3^n + 8 \cdot 2^n - 5(-1)^n & 3 \cdot 3^n - 4 \cdot 2^n + (-1)^n \\ -18 \cdot 3^n + 24 \cdot 2^n - 6(-1)^n & \dots & \dots \\ -54 \cdot 3^n + 48 \cdot 2^n + 6(-1)^n & \dots & \dots \end{bmatrix}$$

Systems of differential equations

Review. Check out Examples 59 and 60 again. Below we will repeat the same steps, replacing recurrences with differential equations as well as λ^n with $e^{\lambda x}$.

Example 63. Write the (second-order) initial value problem $y'' = y' + 2y$, $y(0) = 0$, $y'(0) = 1$ as a first-order system.

Solution. If $\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, then $\mathbf{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ y' + 2y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{y}$ with $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

This is exactly how we proceeded in Example 59.

Homework. Solve this IVP to find $y(x) = \frac{1}{3}(e^{2x} - e^{-x})$. Then compare with the next example.

Example 64. (preview) Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

- (a) Determine the general solution to $\mathbf{y}' = M\mathbf{y}$.
- (b) Determine a fundamental matrix solution to $\mathbf{y}' = M\mathbf{y}$.
- (c) Solve $\mathbf{y}' = M\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution. In Example 60, we only need to replace 2^n by e^{2x} (root 2) and $(-1)^n$ by e^{-x} (root -1)!

(a) The general solution is $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-x}$.

(b) A fundamental matrix solution is $\Phi(x) = \begin{bmatrix} e^{2x} & -e^{-x} \\ 2 \cdot e^{2x} & e^{-x} \end{bmatrix}$.

(c) $\mathbf{y}(x) = \frac{1}{3} \begin{bmatrix} e^{2x} - e^{-x} \\ 2 \cdot e^{2x} + e^{-x} \end{bmatrix}$

Preview. The special fundamental matrix M^n will be replaced by e^{Mx} , the **matrix exponential**.

Example 65. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. If $\mathbf{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$, then $\mathbf{y}' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ 3y'' - 2y' + y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \mathbf{y}$.

For short, $\mathbf{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \mathbf{y}$

Comment. This is one reason why we care about systems of DEs, even if we work with just one function.

Example 66. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 & y_1(0) &= 2, \quad y_1'(0) = 3, \quad y_2(0) = -1, \quad y_2'(0) = 1 \end{aligned}$$

Write it as a first-order initial value problem in the form $\mathbf{y}' = M\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$.

Solution. If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix}$, then $\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ 2y_1' - 3y_2' + 7y_2 \\ 4y_1' + y_2' - 5y_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \mathbf{y}$.

For short, the system translates into $\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \mathbf{y}$ with $\mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$.