Review. The recurrence $a_{n+1} = 5a_n$ has general solution $a_n = C \cdot 5^n$.

In operator form, the recurrence is $(N-5)a_n = 0$, where $p(N) = N - 5$ is the characteristic polynomial. The characteristic root 5 corresponds to the solution 5^n.

This is analogous to the case of DEs $p(D)y = 0$ where a root r of $p(D)$ corresponds to the solution e^{rx}.

Example 45. ("warmup") Find the general solution to the recursion $a_{n+2} = 4a_{n+1} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N + 4$ has roots $2, 2$.

So a solution is 2^n and, from our discussion of DEs, it is probably not surprising that a second solution is $n \cdot 2^n$.

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n = (C_1 + C_2 n) \cdot 2^n$.

Comment. This is analogous to $(D-2)^2 y' = 0$ having the general solution $y(x) = (C_1 + C_2 x)e^{2x}$.

Check! Let's check that $a_n = n \cdot 2^n$ indeed satisfies the recursion $(N-2)^2 a_n = 0$.

$(N-2)n \cdot 2^n = (n+1)2^{n+1} - 2n \cdot 2^n = 2^{n+1}$, so that $(N-2)^2 n \cdot 2^n = (N-2)2^{n+1} = 0$.

Combined, we obtain the following analog of Theorem 20 for recurrence equations (RE):

Comment. Sequences that are solutions to such recurrences are called **constant recursive** or ***C*-finite**.

Theorem 46. Consider the homogeneous linear RE with constant coefficients $p(N)a_n = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the RE are given by $n^j r^n$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

Moreover. If r is the sole largest root by absolute value among the roots contributing to a_n, then $a_n \approx C r^n$ (if r is not repeated—what if it is?) for large n. In particular, it follows that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r.$$

Advanced comment. Things can get weird if there are several roots of the same absolute value. Consider, for instance, the case $a_n = 2^n + (-2)^n$. Can you see that, in this case, the limit $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ doesn't even exist?

Example 47. Find the general solution to the recursion $a_{n+3} = 2a_{n+2} + a_{n+1} - 2a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 2N^2 - N + 2$ has roots $2, 1, -1$. (Here, we may use some help from a computer algebra system to find the roots.)

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 + C_3 \cdot (-1)^n$.

Example 48. Find the general solution to the recursion $a_{n+3} = 3a_{n+2} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 3N^2 + 4$ has roots $2, 2, -1$. (Again, we may use some help from a computer algebra system to find the roots.)

Hence, the general solution is $a_n = (C_1 + C_2 n) \cdot 2^n + C_3 \cdot (-1)^n$.

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Theorem 49. (Binet's formula) $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof. The recursion $F_{n+1} = F_n + F_{n-1}$ can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 1$ has roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Hence, $F_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $F_0 = C_1 + C_2 \overset{!}{=} 0$, $F_1 = C_1 \cdot \frac{1+\sqrt{5}}{2} + C_2 \cdot \frac{1-\sqrt{5}}{2} \overset{!}{=} 1$.

Solving, we find $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$ so that, in conclusion, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, as claimed. □

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}}\lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \operatorname{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \overset{n\to\infty}{\longrightarrow} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

In fact, it follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Example 50. Consider the sequence a_n defined by $a_{n+2} = 4a_{n+1} + 9a_n$ and $a_0 = 1$, $a_1 = 2$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N - 9$ has roots $\frac{4 \pm \sqrt{52}}{2} \approx 5.6056$, -1.6056. Both roots have to be involved in the solution in order to get integer values.
We conclude that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 + \sqrt{13} \approx 5.6056$ (because $|5.6056| > |-1.6056|$).

Example 51. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$
These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1}F_n$. Can you prove this directly from the recursions? Alternatively, this follows from comparing the Binet-like formulas.

Solution. Proceeding as in the previous example, we find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet-like formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

Crash course: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that, for the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \iff\ & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \iff\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

(b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

Example 52. Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

Solution. The characteristic polynomial is:

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{bmatrix}\right) = (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$

Hence, the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

- To find an eigenvector for $\lambda = 3$, we need to solve $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.
- To find an eigenvector for $\lambda = -2$, we need to solve $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Check! $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

On the other hand, a random other vector like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector: $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Example 53. (homework) Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix}$.

Solution. (final answer only) $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$, and $\boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

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