Example 34. (review) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part roots are $3 \pm 2i, \pm i, \pm i$. Hence, there has to be a particular solution of the form
$y_p = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x) + (C_3 + C_4 x)\cos(x) + (C_5 + C_6 x)\sin(x)$.

Continuing to find a particular solution. To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

Sage

In practice, we are happy to let a machine do tedious computations. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at sagemath.org. Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at cocalc.com from any browser.

[For basic computations, you can also simply use the textbox on our course website.]

Sage is built as a **Python** library, so any Python code is valid. For starters, we will use it as a fancy calculator.

Example 35. To solve the differential equation $y'' + 4y' + 4y = 7e^{-2x}$, as we did in Example 29, we can use the following:

```
>>> x = var('x')
>>> y = function('y')(x)
>>> desolve(diff(y,x,2) + 4*diff(y,x) + 4*y == 7*exp(-2*x), y)
```

$$\frac{7}{2}x^2 e^{(-2x)} + (K_2 x + K_1) e^{(-2x)}$$

This confirms, as we had found, that the general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 36. Similarly, Sage can solve initial value problems such as $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

```
>>> x = var('x')
>>> y = function('y')(x)
>>> desolve(diff(y,x,2) - diff(y,x) - 2*y == 0, y, ics=[0,4,5])
```

$$3e^{(2x)} + e^{(-x)}$$

This matches the (unique) solution $y(x) = 3e^{2x} + e^{-x}$ that we derived in Example 18.

Higher order. Unfortunately, the command desolve currently only works like this for differential equations of first and second order. To likewise solve a third-order differential equation, we can use the function desolve_laplace instead. For instance, to solve the IVP $y''' = 3y'' - 4y$ with $y(0) = 1$, $y'(0) = -2$, $y''(0) = 3$, use

```
>>> desolve_laplace(diff(y,x,3) == 3*diff(y,x,2) - 4*y, y, ics=[0,1,-2,3])
```

$$x e^{(2x)} - \frac{2}{3}e^{(2x)} + \frac{5}{3}e^{(-x)}$$

to find that the unique solution is $y(x) = \frac{1}{3}(3x - 2)e^{2x} + \frac{5}{3}e^{-x}$.

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More on differential operators

Example 37. We have been factoring differential operators like $D^2+4D+4=(D+2)^2$.

Things become much more complicated when the coefficients are not constant!

For instance, the linear DE $y''+4y'+4xy=0$ can be written as $Ly=0$ with $L=D^2+4D+4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $xD\neq Dx$!!

Indeed, $(xD)f(x)=xf'(x)$ while $(Dx)f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)]=f(x)+xf'(x)=(1+xD)f(x)$.

This computation shows that, in fact, $Dx=xD+1$.

Review. Linear DEs are those that can be written as $Ly=f(x)$ where L is a linear differential operator: namely,

$$L=p_n(x)D^n+p_{n-1}(x)D^{n-1}+\ldots+p_1(x)D+p_0(x). \tag{1}$$

Recall that the operators xD and Dx are not the same: instead, $Dx=xD+1$.

We say that an operator of the form (1) is in **normal form**.

For instance. xD is in normal form, whereas Dx is not in normal form. It follows from the previous example that the normal form of Dx is $xD+1$.

Example 38. Let $a=a(x)$ be some function.

(a) Write the operator Da in normal form [normal form means as in (1)].

(b) Write the operator D^2a in normal form.

Solution.

(a) $(Da)f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[a(x)\,f(x)]=a'(x)f(x)+a(x)f'(x)=(a'+aD)f(x)$
Hence, $Da=aD+a'$.

(b) $(D^2a)f(x)=\frac{\mathrm{d}^2}{\mathrm{d}x^2}[a(x)\,f(x)]=\frac{\mathrm{d}}{\mathrm{d}x}[a'(x)f(x)+a(x)f'(x)]=a''(x)f(x)+2a'(x)f'(x)+a(x)f''(x)$
$=(a''+2a'D+aD^2)f(x)$
Hence, $D^2a=aD^2+2a'D+a''$.
Alternatively. We can also use $Da=aD+a'$ from the previous part and work with the operators directly:
$D^2a=D(Da)=D(aD+a')=DaD+Da'=(aD+a')D+a'D+a''=aD^2+2a'D+a''$.

Example 39. Suppose that a and b depend on x. Expand $(D+a)(D+b)$ in normal form.

Solution. $(D+a)(D+b)=D^2+Db+aD+ab=D^2+(bD+b')+aD+ab=D^2+(a+b)D+ab+b'$

Comment. Of course, if b is a constant, then $b'=0$ and we just get the familiar expansion.

Comment. At this point, it is not surprising that, in general, $(D+a)(D+b)\neq(D+b)(D+a)$.

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Example 40. Suppose we want to factor $D^2 + pD + q$ as $(D+a)(D+b)$. [p, q, a, b depend on x]

(a) Spell out equations to find a and b.

(b) Find all factorizations of D^2. [An obvious one is $D^2 = D \cdot D$ but there are others!]

Solution.

(a) Matching coefficients with $(D+a)(D+b) = D^2 + (a+b)D + ab + b'$ (we expanded this in the previous example), we find that we need

$$p = a + b, \quad q = ab + b'.$$

Equivalently, $a = p - b$ and $q = (p-b)b + b'$. The latter is a nonlinear (!) DE for b. Once solved for b, we obtain a as $a = p - b$.

(b) This is the case $p = q = 0$. The DE for b becomes $b' = b^2$.
Because it is separable (show all details!), we find that $b(x) = \frac{1}{C - x}$ or $b(x) = 0$.

Since $a = -b$, we obtain the factorizations $D^2 = \left(D - \frac{1}{C-x}\right)\left(D + \frac{1}{C-x}\right)$ and $D^2 = D \cdot D$.

Our computations show that there are no further factorizations.

Comment. Note that this example illustrates that factorization of differential operators is not unique!

For instance, $D^2 = D \cdot D$ and $D^2 = \left(D + \frac{1}{x}\right) \cdot \left(D - \frac{1}{x}\right)$ (the case $C = 0$ above).

Comment. In general, the nonlinear DE for b does not have any polynomial or rational solution (or, in fact, any solution that can be expressed in terms of functions that we are familiar with).