Midterm #2 – Practice

MATH 332 — Differential Equations II
Midterm: Friday, Apr 11, 2025

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. Determine the equilibrium points of the system $\frac{dx}{dt} = (x^2-4)y$, $\frac{dy}{dt} = x^2y - 3xy + 5$ and classify their stability.

Solution. To find the equilibrium points, we solve $(x^2-4)y = 0$ and $x^2y - 3xy + 5 = 0$. The first equation implies that we have $x = \pm 2$, or $y = 0$.

- If $x = 2$, then the second equation becomes $-2y + 5 = 0$ which implies $y = \frac{5}{2}$.
- If $x = -2$, then the second equation becomes $10y + 5 = 0$ which implies $y = -\frac{1}{2}$.
- If $y = 0$, then the second equation becomes $5 = 0$ which has no solution.

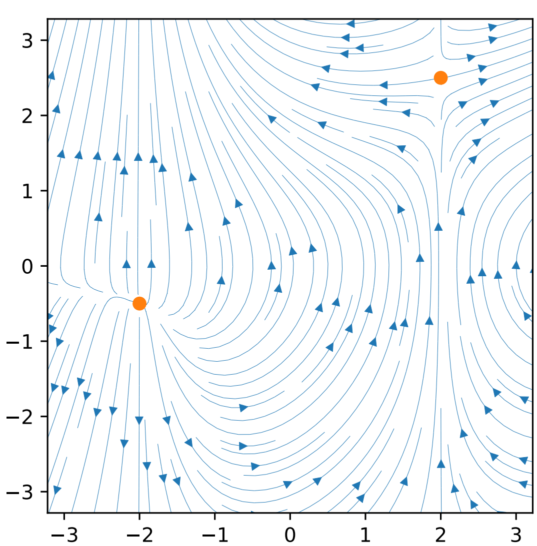
We conclude that we have two equilibrium points: $\left(2, \frac{5}{2}\right)$, $\left(-2, -\frac{1}{2}\right)$.

Our system is $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ with $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} (x^2-4)y \\ x^2y - 3xy + 5 \end{bmatrix}$.

The Jacobian matrix is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2xy & x^2-4 \\ 2xy - 3y & x^2 - 3x \end{bmatrix}$.

- At $\left(2, \frac{5}{2}\right)$, the Jacobian matrix is $J\left(2, \frac{5}{2}\right) = \begin{bmatrix} 10 & 0 \\ \frac{5}{2} & -2 \end{bmatrix}$. We can read off that the eigenvalues are $10, -2$ (because the matrix is triangular). Since one is positive and the other is negative, $\left(2, \frac{5}{2}\right)$ is a saddle. In particular, $\left(2, \frac{5}{2}\right)$ is unstable.
- At $\left(-2, -\frac{1}{2}\right)$, the Jacobian matrix is $J\left(-2, -\frac{1}{2}\right) = \begin{bmatrix} 2 & 0 \\ \frac{7}{2} & 10 \end{bmatrix}$. We can read off that the eigenvalues are $2, 10$. Since both are positive, $\left(-2, -\frac{1}{2}\right)$ is a nodal source. In particular, $\left(-2, -\frac{1}{2}\right)$ is unstable.

The following phase portrait confirms our analysis:



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Problem 2. Let $y(x)$ be the unique solution to the IVP $y''=x+2y^3$, $y(0)=1$, $y'(0)=2$.

Determine the first several terms (up to x^4) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0)=0+2y(0)^3=2$.

Differentiating both sides of the DE, we obtain $y'''=1+6y^2\,y'$. In particular, $y'''(0)=13$.

Continuing, $y^{(4)}=12y(y')^2+6y^2y''$ so that $y^{(4)}(0)=12\cdot 1\cdot 2^2+6\cdot 1^2\cdot 2=60$.

Hence, $y(x)=y(0)+y'(0)x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\frac{1}{24}y^{(4)}(0)x^4+\ldots=1+2x+x^2+\frac{13}{6}x^3+\frac{5}{2}x^4+\ldots$

Solution. (plug in power series) Taking into account the initial conditions, $y=1+2x+a_2x^2+a_3x^3+a_4x^4+\ldots$.

Therefore, $y''=2a_2+6a_3x+12a_4x^2+\ldots$

On the other hand, $y^3=1+6x+(3a_2+12)x^2+\ldots$

Equating coefficients of y'' and $x+2y^3$, we find $2a_2=2$, $6a_3=1+2\cdot 6=13$, $12a_4=2(3a_2+12)$.

So $a_2=1$, $a_3=\frac{13}{6}$, $a_4=\frac{1}{2}a_2+2=\frac{5}{2}$ and, hence, $y(x)=1+2x+x^2+\frac{13}{6}x^3+\frac{5}{2}x^4+\ldots$

Problem 3. Consider the DE $y''=x(x^2+7)\,y'+(x^2+3)y$.

Derive a recursive description of a power series solution $y(x)$ (around $x=0$).

Solution. Let us spell out the power series for y, x^2y, xy', x^3y', y'':

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$x^2y(x)=\sum_{n=0}^{\infty}a_nx^{n+2}=\sum_{n=2}^{\infty}a_{n-2}x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}$$

$$xy'(x)=\sum_{n=1}^{\infty}na_nx^n$$

$$x^3y'(x)=\sum_{n=1}^{\infty}na_nx^{n+2}=\sum_{n=3}^{\infty}(n-2)a_{n-2}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Hence, the DE becomes:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=\sum_{n=3}^{\infty}(n-2)a_{n-2}x^n+7\sum_{n=1}^{\infty}na_nx^n+\sum_{n=2}^{\infty}a_{n-2}x^n+3\sum_{n=0}^{\infty}a_nx^n.$$

We compare coefficients of x^n:

- $n=0$: $2a_2=3a_0$, so that $a_2=\frac{3}{2}a_0$.
- $n=1$: $6a_3=7a_1+3a_1$, so that $a_3=\frac{5}{3}a_1$.
- $n=2$: $12a_4=14a_2+a_0+3a_2$, so that $a_4=\frac{1}{12}a_0+\frac{17}{12}a_2=\frac{1}{12}a_0+\frac{17}{12}\frac{3}{2}a_0=\frac{53}{24}a_0$.
- $n\geqslant 3$: $(n+2)(n+1)a_{n+2}=(n-2)a_{n-2}+7na_n+a_{n-2}+3a_n$

  $a_{n+2}=\frac{7n+3}{(n+2)(n+1)}a_n+\frac{n-1}{(n+2)(n+1)}a_{n-2}$.

Equivalently, for $n \geqslant 5$, $a_n = \frac{7n-11}{n(n-1)} a_{n-2} + \frac{n-3}{n(n-1)} a_{n-4}$.

In conclusion, the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is recursively determined by

$$a_2 = \frac{3}{2} a_0, \quad a_3 = \frac{5}{3} a_1, \quad a_4 = \frac{53}{24} a_0, \quad a_n = \frac{7n-11}{n(n-1)} a_{n-2} + \frac{n-3}{n(n-1)} a_{n-4} \quad \text{for } n \geqslant 5.$$

(The values a_0 and a_1 are the initial conditions.)

Comment. The formula for a_n also holds for $n=4$. Can you see why?

Problem 4. Find a minimum value for the radius of convergence of a power series solution to $(4x^2+1)y'' = \frac{3y'-y}{x+1}$ at $x=3$.

Solution. Note that this is a linear DE! (Otherwise, we could not proceed.) Rewriting the DE as $y'' - \frac{3}{(x+1)(4x^2+1)} y' + \frac{1}{(x+1)(4x^2+1)} y = 0$, we see that the singular points are $x = \pm i/2, -1$.

Note that $x=3$ is an ordinary point of the DE and that the distance to the nearest singular point is $|3 - (\pm i/2)| = \sqrt{3^2 + (1/2)^2} = \frac{1}{2}\sqrt{37} \approx 3.04$ (the distance to -1 is $|3-(-1)| = 4$).

Hence, the DE has power series solutions about $x=3$ with radius of convergence at least $\frac{1}{2}\sqrt{37}$.

Problem 5. Spell out the power series (around $x=0$) of the following functions.

(a) e^{-3x}

(b) $\sin(3x^2)$

(c) $\frac{5}{1+7x^2}$

Solution.

(a) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n$.

(b) Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, we have $\sin(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{4n+2}$.

(c) Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{5}{1+7x^2} = 5\sum_{n=0}^{\infty} (-7x^2)^n = 5\sum_{n=0}^{\infty} (-7)^n x^{2n}$.

Problem 6.

(a) Suppose $y(x)$ has the power series $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$. How can we compute the a_n from $y(x)$?

(b) Suppose $f(t)$ has the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right)$.

How can we compute the a_n and b_n from $f(t)$?

Solution.

(a) $a_n = \frac{y^{(n)}(x_0)}{n!}$

(b) The Fourier coefficients a_n, b_n can be computed as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t.$$

Problem 7. Consider the function $f(t) = 2(1-t)$, defined for $t \in [0,1]$.

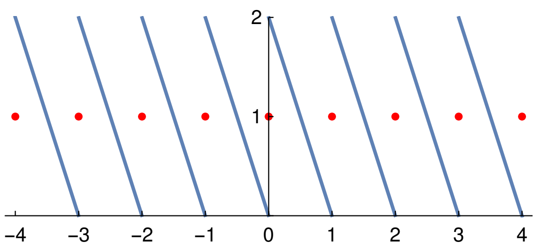
(a) Sketch the Fourier series of $f(t)$ for $t \in [-4,4]$.

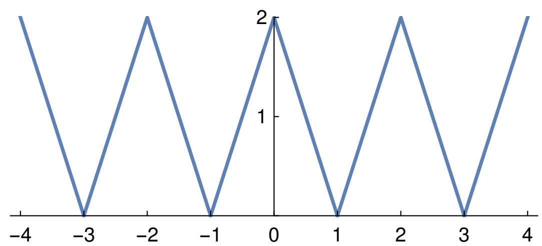
(b) Sketch the Fourier cosine series of $f(t)$ for $t \in [-4,4]$.

(c) Sketch the Fourier sine series of $f(t)$ for $t \in [-4,4]$.

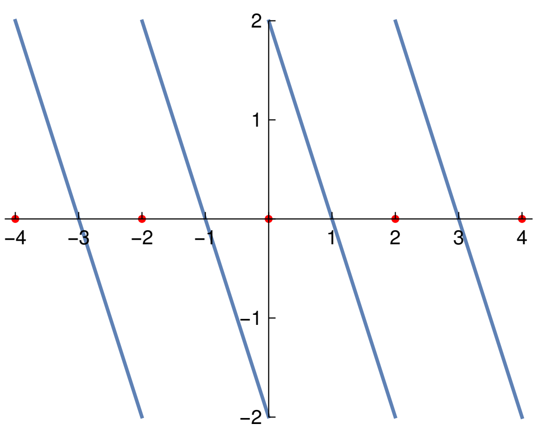
In each sketch, carefully mark the values of the Fourier series at discontinuities.

Solution. The Fourier series (i.e. the 1-periodic extension) as well as the Fourier cosine series (i.e. the 2-periodic even extension):





The Fourier sine series (i.e. the 2-periodic odd extension):



In each sketch, the function values at discontinuities are marked in red.

Problem 8. A mass-spring system is described by the DE $my'' + 7y = F(t)$ where $F(t)$ is an external force with period 3. For which values of m can resonance occur?

Solution. $F(t)$ has a Fourier series of the form $F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2\pi n t}{3}\right) + b_n\sin\left(\frac{2\pi n t}{3}\right)\right)$.

The roots of $p(D) = mD^2 + 7$ are $\pm i\sqrt{\frac{7}{m}}$, so that the natural frequency is $\sqrt{\frac{7}{m}}$. Resonance therefore can occur if $\sqrt{\frac{7}{m}} = \frac{2\pi n}{3}$ for some $n \in \{1,2,3,\ldots\}$. Equivalently, resonance can occur if $m = \frac{63}{4\pi^2 n^2}$ for some $n \in \{1,2,3,\ldots\}$.

Problem 9. A mass-spring system is described by the equation

$$my'' + y = F(t),$$

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where the external force has the Fourier series $F(t) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\sin\left(\frac{nt}{3}\right)$.

(a) For which m does resonance occur?

(b) Find the general solution when $m = 1/9$.

Solution.

(a) The roots of $p(D) = mD^2 + 1$ are $\pm i/\sqrt{m}$, so that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m} = n/3$ for some $n \in \{1, 3, 5, \ldots\}$. Equivalently, resonance occurs if $m = 9/n^2$ for an odd integer $n \geqslant 1$ (that is, $m = 9, 1, 9/25, 9/49, \ldots$).

(b) In this case, the natural frequency is 3 and we have resonance because $3 = n/3$ for $n = 9$. For $n \neq 9$ we solve

$$\frac{1}{9}y'' + y = \frac{1}{n^2}\sin\left(\frac{nt}{3}\right).$$

This has a solution of the form $y_p = A\cos\left(\frac{nt}{3}\right) + B\sin\left(\frac{nt}{3}\right)$ where A, B are undetermined. Plugging into the DE:

$$\frac{1}{9}y_p'' + y_p = A\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\cos\left(\frac{nt}{3}\right) + B\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\sin\left(\frac{nt}{3}\right) \overset{!}{=} \frac{1}{n^2}\sin\left(\frac{nt}{3}\right)$$

It follows that $A = 0$ (we could have seen that coming...) and

$$B = \frac{1}{n^2\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)} = \frac{81}{n^2(81 - n^2)}, \qquad y_p = \frac{81}{n^2(81 - n^2)}\sin\left(\frac{nt}{3}\right).$$

The case $n = 9$ has to be done separately: because of resonance there now exists a solution of the form

$$y_p = At\cos(3t) + Bt\sin(3t).$$

Plugging into the DE:

$$\frac{1}{9}y_p'' + y_p = \frac{2}{3}B\cos(3t) - \frac{2}{3}A\sin(3t) \overset{!}{=} \frac{1}{81}\sin(3t)$$

It follows that $B = 0$ and $A = -\frac{1}{54}$. So $y_p = -\frac{1}{54}t\cos(3t)$. By superposition it follows that

$$\frac{1}{9}y'' + y = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\sin\left(\frac{nt}{3}\right) \quad \text{has solution} \quad y_p = -\frac{1}{54}t\cos(3t) + \sum_{\substack{n=1\\ n \text{ odd},\, n \neq 9}} \frac{81}{n^2(81 - n^2)}\sin\left(\frac{nt}{3}\right).$$

The general solution is $y(t) = y_p(t) + A\cos(3t) + B\sin(3t)$.

Problem 10. Derive a recursive description of the power series (around $x = 0$) for $y(x) = \frac{1}{1 - 2x - 5x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1 = (1 - 2x - 5x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty} a_n x^{n+1} - 5\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1} x^n - 5\sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

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We compare coefficients of x^n:

- $n=0$: $1=a_0$.
- $n=1$: $0=a_1-2a_0$, so that $a_1=2a_0=2$.
- $n\geqslant 2$: $0=a_n-2a_{n-1}-5a_{n-2}$ or, equivalently, $a_n=2a_{n-1}+5a_{n-2}$.

In conclusion, the power series $\frac{1}{1-2x-5x^2}=\sum_{n=0}^{\infty} a_n x^n$ is recursively determined by

$$a_0=1, \quad a_1=2, \quad a_n=2a_{n-1}+5a_{n-2} \quad \text{for } n\geqslant 2.$$

Problem 11. Compute the Fourier sine series of the function $f(t)$, defined for $t\in(0,L)$, with $f(t)=3$.

Solution. The odd $2L$-periodic extension of $f(t)$ takes the values $f(t)=\begin{cases} -3, & \text{for } t\in(-L,0).\\ +3, & \text{for } t\in(0,L).\end{cases}$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t=\frac{2}{L}\int_0^L 3\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t=\frac{6}{L}\left[-\frac{L}{\pi n}\cos\left(\frac{n\pi t}{L}\right)\right]_0^L=\frac{6}{\pi n}[1-\cos(n\pi)]\\ &= \frac{6}{\pi n}[1-(-1)^n]=\begin{cases} \frac{12}{\pi n}, & \text{if } n \text{ is odd,}\\ 0, & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

Thus the Fourier sine series is:

$$\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{12}{\pi n}\sin\left(\frac{n\pi t}{L}\right)$$

Problem 12. Suppose that the matrix A satisfies $e^{Ax}=\frac{1}{7}\begin{bmatrix} e^{-9x}+6e^{-2x} & -2e^{-9x}+2e^{-2x}\\ -3e^{-9x}+3e^{-2x} & 6e^{-9x}+e^{-2x}\end{bmatrix}$.

(a) Solve $\boldsymbol{y}'=A\boldsymbol{y}$, $\boldsymbol{y}(0)=\begin{bmatrix} 2\\ 1\end{bmatrix}$.

(b) Solve $\boldsymbol{y}'=A\boldsymbol{y}+\begin{bmatrix} 0\\ 3e^x\end{bmatrix}$, $\boldsymbol{y}(0)=\begin{bmatrix} 2\\ 1\end{bmatrix}$.

(c) What is A?

Solution.

(a) $\boldsymbol{y}(x)=e^{Ax}\begin{bmatrix} 2\\ 1\end{bmatrix}=\frac{1}{7}\begin{bmatrix} 14e^{-2x}\\ 7e^{-2x}\end{bmatrix}=\begin{bmatrix} 2e^{-2x}\\ e^{-2x}\end{bmatrix}$

(b) $\boldsymbol{y}(x)=e^{Ax}\begin{bmatrix} 2\\ 1\end{bmatrix}+e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$. We compute:

$\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t=\int_0^x \frac{1}{7}\begin{bmatrix} e^{9t}+6e^{2t} & -2e^{9t}+2e^{2t}\\ -3e^{9t}+3e^{2t} & 6e^{9t}+e^{2t}\end{bmatrix}\begin{bmatrix} 0\\ 3e^t\end{bmatrix}\mathrm{d}t=\frac{3}{7}\int_0^x\begin{bmatrix} -2e^{10t}+2e^{3t}\\ 6e^{10t}+e^{3t}\end{bmatrix}\mathrm{d}t=\frac{3}{7}\begin{bmatrix} -\frac{1}{5}e^{10x}+\frac{2}{3}e^{3x}-\frac{7}{15}\\ \frac{3}{5}e^{10x}+\frac{1}{3}e^{3x}-\frac{14}{15}\end{bmatrix}$
$=\frac{1}{35}\begin{bmatrix} -3e^{10x}+10e^{3x}-7\\ 9e^{10x}+5e^{3x}-14\end{bmatrix}$

Hence, $e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t=\frac{1}{7}\begin{bmatrix} e^{-9x}+6e^{-2x} & -2e^{-9x}+2e^{-2x}\\ -3e^{-9x}+3e^{-2x} & 6e^{-9x}+e^{-2x}\end{bmatrix}\frac{1}{35}\begin{bmatrix} -3e^{10x}+10e^{3x}-7\\ 9e^{10x}+5e^{3x}-14\end{bmatrix}=\frac{1}{35}\begin{bmatrix} 3e^{-9x}-10e^{-2x}+7e^x\\ -9e^{-9x}-5e^{-2x}+14e^x\end{bmatrix}$.

Finally, $\boldsymbol{y}(x)=\begin{bmatrix} 2e^{-2x}\\ e^{-2x}\end{bmatrix}+\frac{1}{35}\begin{bmatrix} 3e^{-9x}-10e^{-2x}+7e^x\\ -9e^{-9x}-5e^{-2x}+14e^x\end{bmatrix}=\frac{1}{35}\begin{bmatrix} 3e^{-9x}+60e^{-2x}+7e^x\\ -9e^{-9x}+30e^{-2x}+14e^x\end{bmatrix}$.

(c) Replacing e^{-9x} with $(-9)^n$ as well as e^{-2x} with $(-2)^n$, we conclude $A^n=\frac{1}{7}\begin{bmatrix} (-9)^n+6\cdot(-2)^n & -2\cdot(-9)^n+2\cdot(-2)^n\\ -3\cdot(-9)^n+3\cdot(-2)^n & 6\cdot(-9)^n+(-2)^n\end{bmatrix}$.

In particular, $A=\begin{bmatrix} -3 & 2\\ 3 & -8\end{bmatrix}$.

Alternatively. Like any fundamental matrix, $\Phi=e^{Ax}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax}=Ae^{Ax}$.

Hence, $A=\left[\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax}\right]_{x=0}=\left[\frac{1}{7}\begin{bmatrix} -9e^{-9x}-12e^{-2x} & 18e^{-9x}-4e^{-2x}\\ 27e^{-9x}-6e^{-2x} & -54e^{-9x}-2e^{-2x}\end{bmatrix}\right]_{x=0}=\begin{bmatrix} -3 & 2\\ 3 & -8\end{bmatrix}$.

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