Example 176. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 1 \\ u(x,2) &= 0 \qquad \text{(BC)} \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned}$$

Solution. This is the special case of the previous example with $a=1$, $b=2$ and $f(x)=1$ for $x\in(0,1)$.

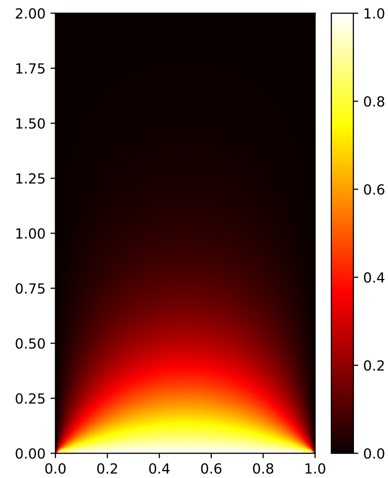
From Example 136, we know that $f(x)$ has the Fourier sine series

$$f(x)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi x),\qquad x\in(0,1).$$

Hence,

$$u(x,y)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{\pi n(4-y)}).$$

Comment. The temperature at the center is $u(\frac{1}{2},1)\approx 0.0549$ (only the first term of the infinite sum suffices for this estimate; the first three terms suffice for 9 digits of accuracy).



Example 177. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 0 \\ u(x,2) &= 3 \qquad \text{(BC)} \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned}$$

Solution. Instead of starting from scratch (homework exercise!), let us reuse our computations:

Let $v(x,y)=u(x,2-y)$. Then $v_{xx}+v_{yy}=0$, $v(x,0)=3$, $v(x,2)=0$, $v(0,y)=0$, $v(1,y)=0$.

Hence, it follows from the previous example that

$$v(x,y)=3\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{\pi n(4-y)}).$$

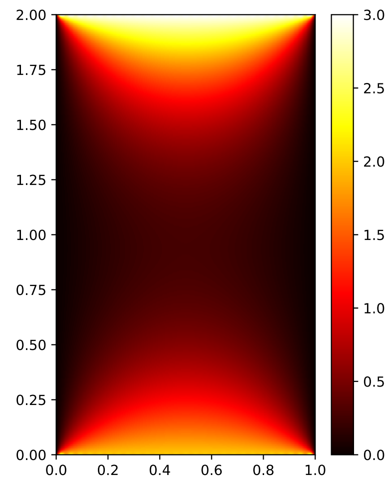
Consequently,

$$u(x,y)=v(x,2-y)=3\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n(2-y)}-e^{\pi n(2+y)}).$$

Armin Straub
straub@southalabama.edu

Example 178. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0)=2, \quad u(x,2)&=3 \\ u(0,y)=0, \quad u(1,y)&=0 \end{aligned}$$



Solution. Note that $u(x,\ y)$ is a combination of the solutions to the previous two examples!

$$u(x,y) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{\sin(\pi n x)}{1 - e^{4\pi n}} [2(e^{\pi n y} - e^{-\pi n (y-4)}) + 3(e^{\pi n (2-y)} - e^{\pi n (2+y)})].$$

Example 179. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{(PDE)} \\ u(x,0) &= 4\sin(\pi x) - 5\sin(3\pi x) \\ u(x,2) &= 0 && \text{(BC)} \\ u(0,y) &= 0 \\ u(3,y) &= 0 \end{aligned}$$

Solution.

- We look for solutions $u(x,y) = X(x)Y(y)$ (**separation of variables**).
  Plugging into (PDE), we get $X''(x)Y(y) + X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const}$.
  We thus have $X'' - \text{const}\, X = 0$ and $Y'' + \text{const}\, Y = 0$.
- From the last three (BC), we get $X(0) = 0$, $X(3) = 0$, $Y(2) = 0$.
- So X solves $X'' + \lambda X = 0$ (we choose $\lambda = -\text{const}$), $X(0) = 0$, $X(3) = 0$.
  From earlier (or do it!), we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \sin\left(\frac{1}{3}\pi n x\right)$ corresponding to $\lambda = \left(\frac{1}{3}\pi n\right)^2$, $n = 1, 2, 3\ldots$.
- On the other hand, Y solves $Y'' - \lambda Y = 0$, and hence $Y(y) = Ae^{\sqrt{\lambda}\, y} + Be^{-\sqrt{\lambda}\, y}$.
  The condition $Y(2) = 0$ implies that $Ae^{2\sqrt{\lambda}} + Be^{-2\sqrt{\lambda}} = 0$ so that $B = -Ae^{4\sqrt{\lambda}}$.
  Hence, $Y(y) = A(e^{\sqrt{\lambda}\, y} - e^{\sqrt{\lambda}(4-y)}) = A\left(e^{\frac{1}{3}\pi n y} - e^{\frac{1}{3}\pi n (4-y)}\right)$.
- Taken together, we have the solutions $u_n(x,\ y) = \sin\left(\frac{1}{3}\pi n x\right)\left(e^{\frac{1}{3}\pi n y} - e^{\frac{1}{3}\pi n (4-y)}\right)$ solving (PDE)+(BC), with the exception of $u(x,0) = 4\sin(\pi x) - 5\sin(3\pi x)$.
- At $y = 0$, $u_n(x,0) = \sin\left(\frac{1}{3}\pi n x\right)\left(1 - e^{\frac{4}{3}\pi n}\right)$.
  In particular, $u_3(x,0) = \sin(\pi x)(1 - e^{4\pi})$ and $u_9(x,0) = \sin(3\pi x)(1 - e^{12\pi})$.
  Hence, $4\sin(\pi x) - 5\sin(3\pi x) = \frac{4}{1 - e^{4\pi}} u_3(x,0) - \frac{5}{1 - e^{12\pi}} u_9(x,0)$. Therefore our overall solution is

$$\begin{aligned} u(x,y) &= \frac{4}{1 - e^{4\pi}} u_3(x,y) - \frac{5}{1 - e^{12\pi}} u_9(x,y) \\ &= \frac{4}{1 - e^{4\pi}} \sin(\pi x)(e^{\pi y} - e^{\pi(4-y)}) - \frac{5}{1 - e^{12\pi}} \sin(3\pi x)(e^{3\pi y} - e^{3\pi(4-y)}). \end{aligned}$$

Comment. Of course, in general, our inhomogeneous (BC) will be a function $f(x)$ that is not such an obvious combination of our special solutions $u_n(x,0)$. In that case, we need to compute an appropriate Fourier expansion of $f(x)$ first (here, the Fourier sine series of $f(x)$).