Fourier cosine series and Fourier sine series

Suppose we have a function $f(t)$ which is defined on a finite interval $[0, L]$. Depending on the kind of application, we can extend $f(t)$ to a periodic function in three natural ways; in each case, we can then compute a Fourier series for $f(t)$ (which will agree with $f(t)$ on $[0, L]$).

Comment. Here, we do not worry about the definition of $f(t)$ at specific individual points like $t=0$ and $t=L$, or at jump discontinuities. Recall that, at a discontinuity, a Fourier series takes the average value.

(a) We can extend $f(t)$ to an L-periodic function.

In that case, we obtain the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n t}{L}\right) + b_n \sin\left(\frac{2\pi n t}{L}\right)\right)$.

(b) We can extend $f(t)$ to an even $2L$-periodic function.

In that case, we obtain the **Fourier cosine series** $f(t) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos\left(\frac{\pi n t}{L}\right)$.

(c) We can extend $f(t)$ to an odd $2L$-periodic function.

In that case, we obtain the **Fourier sine series** $f(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{\pi n t}{L}\right)$.

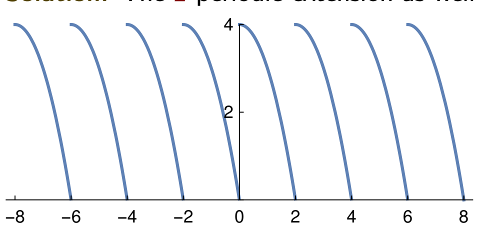
Example 137. Consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

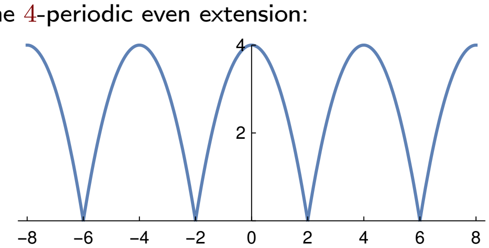
(a) Sketch the 2-periodic extension of $f(t)$.

(b) Sketch the 4-periodic even extension of $f(t)$.

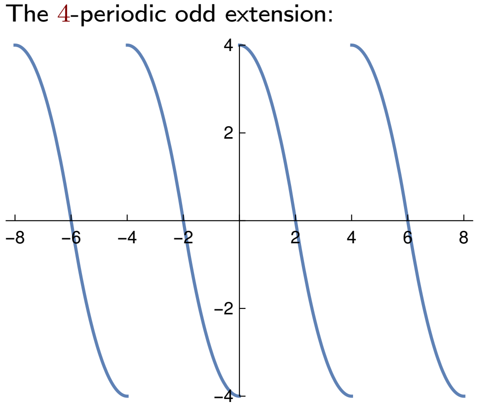
(c) Sketch the 4-periodic odd extension of $f(t)$.

Solution. The 2-periodic extension as well as the 4-periodic even extension:





The 4-periodic odd extension:



Armin Straub
straub@southalabama.edu

Example 138. As in the previous example, consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

(a) Let $F(t)$ be the Fourier series of $f(t)$ (meaning the 2-periodic extension of $f(t)$). Determine $F(2)$, $F\left(\frac{5}{2}\right)$ and $F\left(-\frac{1}{2}\right)$.

(b) Let $G(t)$ be the Fourier cosine series of $f(t)$. Determine $G(2)$, $G\left(\frac{5}{2}\right)$ and $G\left(-\frac{1}{2}\right)$.

(c) Let $H(t)$ be the Fourier sine series of $f(t)$. Determine $H(2)$, $H\left(\frac{5}{2}\right)$ and $H\left(-\frac{1}{2}\right)$.

Solution.

(a) Note that the extension of $f(t)$ has discontinuities at $\ldots, -2, 0, 2, 4, \ldots$ (see plot in previous example) and recall that the Fourier series takes average values at these discontinuities:
$F(2) = \frac{1}{2}(F(2^-) + F(2^+)) = \frac{1}{2}(0 + 4) = 2$
$F\left(\frac{5}{2}\right) = F\left(\frac{5}{2} - 2\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$
$F\left(-\frac{1}{2}\right) = F\left(-\frac{1}{2} + 2\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$

(b) $G(2) = f(2) = 0$ (see plot!)
[Note that $G(2^+) = G(2^+ - 4) = G(-2^+) = G(2^-)$ where we used that G is even in the last step; in fact, we can show like this that the Fourier cosine series of a continuous function is always continuous.]
$G\left(\frac{5}{2}\right) = G\left(\frac{5}{2} - 4\right) = G\left(-\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$
$G\left(-\frac{1}{2}\right) \overset{\text{even}}{=} G\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$

(c) $H(2) = \frac{1}{2}(f(2^-) - f(2^-)) = 0$ (see plot!)
[Note that $H(2^+) = H(2^+ - 4) = H(-2^+) = -H(2^-)$ where we used that H is odd in the last step; in fact, we can show like this that the Fourier sine series of a continuous function is always 0 at the jumps.]
$H\left(\frac{5}{2}\right) = H\left(\frac{5}{2} - 4\right) = H\left(-\frac{3}{2}\right) = -f\left(\frac{3}{2}\right) = -\frac{7}{4}$
$H\left(-\frac{1}{2}\right) \overset{\text{odd}}{=} -H\left(\frac{1}{2}\right) = -f\left(\frac{1}{2}\right) = -\frac{15}{4}$

Differentiating and integrating Fourier series

Theorem 139. If $f(t)$ is **continuous** and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t) = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Caution! We cannot simply differentiate termwise if $f(t)$ is lacking continuity. See the next example.

Comment. On the other hand, we can integrate termwise (going from the Fourier series of $f' = g$ to the Fourier series of $f = \int g$ because the latter will be continuous). This is illustrated in the example after the next.

Example 140. (caution!) The function $g(t) = \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(n\pi t)$ from Example 136 is not continuous. For all values, except the discontinuities, we have $g'(t) = 0$. On the other hand, differentiating the Fourier series termwise, results in $4\sum_{n \text{ odd}} \cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t = 0$). This illustrates that we cannot apply Theorem 139 because $g(t)$ is lacking continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]

Example 141. Let $h(t)$ be the 2-periodic function with $h(t) = |t|$ for $t \in [-1, 1]$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n = 0$. Computing a_n is left as an exercise.

Solution. Note that $h(t) = \begin{cases} -t & \text{for } t \in (-1, 0) \\ +t & \text{for } t \in (0, 1) \end{cases}$ is continuous and $h'(t) = g(t)$, with $g(t)$ as in Example 136. Hence, we can apply Theorem 139 to conclude

$$h'(t) = g(t) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t) \quad \Longrightarrow \quad h(t) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \left(-\frac{1}{\pi n}\right) \cos(n\pi t) + C,$$

where $C = \frac{a_0}{2} = \frac{1}{2} \int_{-1}^{1} h(t) \mathrm{d}t = \frac{1}{2}$ is the constant of integration. Thus, $h(t) = \frac{1}{2} - \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} \cos(n\pi t)$.

Remark. Note that $t = 0$ in the last Fourier series, gives us $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$. As an exercise, you can try to find from here the fact that $\sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n \geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3), \zeta(5), \ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

Armin Straub
straub@southalabama.edu