Notes for Lecture 24

Wed, 3/26/2025

Power series of familiar functions

(Unless we specify otherwise, power series are meant to be about $x=0$.)

Example 122. Determine the power series for $\cos(x)$ at $x=0$.

Solution. Let $y(x)=\cos(x)$. After computing a few derivatives, we realize that $y^{(2n)}(x)=(-1)^n\cos(x)$ and $y^{(2n+1)}(x)=-(-1)^n\sin(x)$. In particular, $y^{(2n)}(0)=(-1)^n$ and $y^{(2n+1)}(0)=0$. It follows that

$$\cos(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

Comment. Note that the above observations on $y^{(2n)}$ and $y^{(2n+1)}$ simply reflect the fact that $\cos(x)$ is the unique solution to the IVP $y''=-y$, $y(0)=1$, $y'(0)=0$.

Alternatively. We can also deduce the power series via Euler's formula: $e^{ix}=\cos(x)+i\sin(x)$. Since

$$e^{ix}=\sum_{n=0}^{\infty}\frac{(ix)^n}{n!}=\sum_{m=0}^{\infty}\frac{(ix)^{2m}}{(2m)!}+\sum_{m=0}^{\infty}\frac{(ix)^{2m+1}}{(2m+1)!}=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!}+i\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!},$$

we conclude that $\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ and $\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

Example 123. Determine the first several terms in the power series of $\sin(2x^3)$ at $x=0$.

Solution. (direct—unpleasant) If $f(x)=\sin(2x^3)$, then $f'(x)=6x^2\cos(2x^3)$ as well as $f''(x)=12x\cos(2x^3)-36x^4\sin(2x^3)$ and $f'''(x)=12\cos(2x^3)-216x^3\sin(2x^3)+216x^6\cos(2x^3)$.

In particular, $f(0)=0$, $f'(0)=0$, $f''(0)=0$ and $f'''(0)=12$.

It follows that $f(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\ldots=0+0x+0x^2+\frac{12}{3!}x^3+\ldots=2x^3+\ldots$

Solution. (via series for sine) As done in the previous example for $\cos(x)$, we can derive that

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\ldots$$

It follows that

$$\begin{aligned}\sin(2x^3)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x^3)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n+1}}{(2n+1)!}x^{6n+3}\\&=\frac{2^1}{1!}x^3-\frac{2^3}{3!}x^9+\frac{2^5}{5!}x^{15}-\ldots=2x^3-\frac{4}{3}x^9+\frac{4}{15}x^{15}-\ldots\end{aligned}$$

Example 124. The **hyperbolic cosine** $\cosh(x)$ is defined to be the even part of e^x. In other words, $\cosh(x)=\frac{1}{2}(e^x+e^{-x})$. Determine its power series.

Solution. It follows from $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ that $\cosh(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Comment. Note that $\cosh(ix)=\cos(x)$ (because $\cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$).

Comment. The hyperbolic sine $\sinh(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$ is similarly defined to be the odd part of e^x.

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Example 125. Determine a power series for $\frac{1}{1+x^2}$.

Solution. Replace x with $-x^2$ in $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (geometric series!) to get $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

Example 126. Determine a power series for $\arctan(x)$.

Solution. Recall that $\arctan(x) = \int \frac{\mathrm{d}x}{1+x^2} + C$. Hence, we need to integrate $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

It follows that $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$. Since $\arctan(0) = 0$, we conclude that $C = 0$.

Example 127. Determine a power series for $\ln(x)$ around $x = 1$.

Solution. This is equivalent to finding a power series for $\ln(x+1)$ around $x = 0$ (see the final step).

Observe that $\ln(x+1) = \int \frac{\mathrm{d}x}{1+x} + C$ and that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$.

Integrating, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$. Since $\ln(1) = 0$, we conclude that $C = 0$.

Finally, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is equivalent to $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$.

Comment. Choosing $x = 2$ in $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$ results in $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.
The latter is the alternating harmonic sum.
Can you see from the series for $\ln(x)$ why the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges?

Example 128. (error function) Determine a power series for $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$.

Solution. It follows from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$.

Integrating, we obtain $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

Example 129. Determine the first several terms (up to x^5) in the power series of $\tan(x)$.

Solution. Observe that $y(x) = \tan(x)$ is the unique solution to the IVP $y' = 1 + y^2$, $y(0) = 0$.
We can therefore proceed to determine the first few power series coefficients as we did earlier.
That is, we plug $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ into the DE. Note that $y(0) = 0$ means $a_0 = 0$.

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$$

$$1 + y^2 = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + \ldots)^2 = 1 + a_1^2 x^2 + (2a_1 a_2) x^3 + (2a_1 a_3 + a_2^2) x^4 + \ldots$$

Comparing coefficients, we find: $a_1 = 1$, $2a_2 = 0$, $3a_3 = a_1^2$, $4a_4 = 2a_1 a_2$, $5a_5 = 2a_1 a_3 + a_2^2$.

Solving for $a_2, a_3, \ldots$, we conclude that $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$

Comment. The fact that $\tan(x)$ is an odd function translates into $a_n = 0$ when n is even. If we had realized that at the beginning, our computation would have been simplified.

Advanced comment. The full power series is $\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1}$.

Here, the numbers B_{2n} are (rather mysterious) rational numbers known as **Bernoulli numbers**.
The radius of convergence is $\pi/2$. Note that this is not at all obvious from the DE $y' = 1 + y^2$. This illustrates the fact that nonlinear DEs are much more complicated than linear ones. (There is no analog of Theorem 111.)