**Review.** Theorem 111: If $x_0$ is an ordinary point of a linear IVP, then it is guaranteed to have a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Moreover, its radius of convergence is at least the distance between $x_0$ and the closest singular point.

**Example 115.** Find a minimum value for the radius of convergence of a power series solution to $(x^2+4)y'' - 3xy' + \frac{1}{x+1}y = 0$ at $x = 2$.

**Solution.** The singular points are $x = \pm 2i, -1$. Hence, $x = 2$ is an ordinary point of the DE and the distance to the nearest singular point is $|2 - 2i| = \sqrt{2^2 + 2^2} = \sqrt{8}$ (the distances are $|2 - (-1)| = 3$, $|2 - (\pm 2i)| = \sqrt{8}$). By Theorem 111, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{8}$.

**Example 116. (caution!)** Theorem 111 only holds for linear DEs! For nonlinear DEs, it is very hard to predict whether there is a power series solution and what its radius of convergence is. Consider, for instance, the nonlinear DE $y' - y^2 = 0$.

Its coefficients have no singularities. A solution to this DE is $y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (see Example 119), which clearly has a problem at $x = 1$ (the radius of convergence is $1$).

**On the other hand.** $y(x)$ also solves the linear DE $(1-x)y' - y = 0$ (or, even simpler, the order $0$ "differential" equation $(1-x)y = 1$). Note how the DE has the singular point $x = 1$. Theorem 111 then allows us to predict that $y(x)$ must have a power series with radius of convergence at least $1$.

**Example 117. (Bessel functions)** Consider the DE $x^2y'' + xy' + x^2y = 0$. Derive a recursive description of a power series solutions $y(x)$ at $x = 0$.

**Caution!** Note that $x = 0$ is a singular point (the only) of the DE. Theorem 111 therefore does not guarantee a basis of power series solutions. [However, $x = 0$ is what is called a **regular singular point**; for these, we are guaranteed one power series solution, as well as additional solutions expressed using logarithms and power series.]

**Comment.** We could divide the DE by $x$ (but that wouldn't really change the computations below). The reason for not dividing that $x$ is that this DE is the special case $\alpha = 0$ of the **Bessel equation** $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ (for which no such dividing is possible).

**Solution. (plug in power series)** Let us spell out power series for $x^2y, xy', x^2y''$ starting with $y(x) = \sum_{n=0}^{\infty} a_n x^n$:

$$x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$xy'(x) = \sum_{n=1}^{\infty} n a_n x^n \qquad \text{(because } y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}\text{)}$$

$$x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^n \qquad \text{(because } y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\text{)}$$

Hence, the DE becomes $\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$. We compare coefficients of $x^n$:

- $n = 1$: $a_1 = 0$
- $n \geqslant 2$: $n(n-1)a_n + n a_n + a_{n-2} = 0$, which simplifies to $n^2 a_n = -a_{n-2}$.
  It follows that $a_{2n} = \frac{(-1)^n}{4^n n!^2} a_0$ and $a_{2n+1} = 0$.

**Observation.** The fact that we found $a_1 = 0$ reflects the fact that we cannot represent the general solution through power series alone.

**Comment.** If $a_0 = 1$, the function we found is a **Bessel function** and denoted as $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2}\right)^{2n}$.

The more general Bessel functions $J_\alpha(x)$ are solutions to the DE $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$.

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**Example 118. (caution!)** Consider the linear DE $x^2y' = y - x$. Does it have a convergent power series solution at $x = 0$?

**Important note.** The DE $x^2y' = y - x$ has the singular point $x = 0$. Hence, Theorem 111 does not apply.

**Advanced.** Moreover, in contrast to the previous example, $x = 0$ is not a **regular singular point**. Indeed, as we see below, there is no power series solution of the DE at all.

**Solution.** Let us look for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$x^2y'(x) = x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1)a_{n-1}x^n$$

Hence, $x^2y' = y - x$ becomes $\sum_{n=2}^{\infty} (n-1)a_{n-1}x^n = \sum_{n=0}^{\infty} a_n x^n - x$. We compare coefficients of $x^n$:

- $n = 0$: $a_0 = 0$.
- $n = 1$: $0 = a_1 - 1$, so that $a_1 = 1$.
- $n \geqslant 2$: $(n-1)a_{n-1} = a_n$, from which it follows that $a_n = (n-1)a_{n-1} = (n-1)(n-2)a_{n-2} = \cdots = (n-1)!a_1 = (n-1)!$.

Hence the DE has the "formal" power series solution $y(x) = \sum_{n=1}^{\infty} (n-1)! x^n$.

However, that series is divergent for all $x \neq 0$; that is, the radius of convergence is $0$.

## Inverses of power series

**Example 119. (geometric series)** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

**Why?** If $y(x) = \sum_{n=0}^{\infty} x^n$, then $xy = y - 1$ (write down the power series for both sides!). Hence, $y = \frac{1}{1-x}$.

Alternatively, start with $y = \frac{1}{1-x}$ and note that $y$ solves the order $0$ "differential" (inhomogeneous) equation $(1-x)y = 1$. We can then determine a power series solution as we did in Example 107 to find $y = \sum_{n=0}^{\infty} x^n$.

**Example 120.** Derive a recursive description of the power series for $y(x) = \frac{1}{1-x-x^2}$.

**Solution.** Note that $y(x)$ satisfies the "differential" equation $(1-x-x^2)y = 1$ of order $0$ (as such, we need $0$ initial conditions). We can therefore determine a power series solution as we did in Example 107:

Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1 = (1-x-x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of $x^n$:

- $n = 0$: $1 = a_0$.
- $n = 1$: $0 = a_1 - a_0$, so that $a_1 = a_0 = 1$.
- $n \geqslant 2$: $0 = a_n - a_{n-1} - a_{n-2}$ or, equivalently, $a_n = a_{n-1} + a_{n-2}$.

This is the recursive description of the Fibonacci numbers $F_n$! In particular $a_n = F_n$.

**The first few terms.** $\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$

**Comment.** The function $y(x)$ is said to be a **generating function** for the Fibonacci numbers.

**Challenge.** Can you rederive Binet's formula from partial fractions and the geometric series?

**Example 121. (HW)** Derive a recursive description of the power series for $y(x) = \frac{1+7x}{1-x-2x^2}$.

**Solution.** Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$1 + 7x = (1 - x - 2x^2) \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - 2 \sum_{n=2}^{\infty} a_{n-2} x^n.$$

We compare coefficients of $x^n$:

- $n = 0$: $1 = a_0$.
- $n = 1$: $7 = a_1 - a_0$, so that $a_1 = 7 + a_0 = 8$.
- $n \geqslant 2$: $0 = a_n - a_{n-1} - 2a_{n-2}$.

If we prefer, we can rewrite the final recurrence as $a_{n+2} - a_{n+1} - 2a_n = 0$ for $n \geqslant 0$. The initial conditions are $a_0 = 1$, $a_1 = 8$.

**Comment.** In terms of the recurrence operator $N$, the recurrence is $(N^2 - N - 2)a_n = 0$.

**Comment.** As in Example 46, we can solve this recurrence and obtain a Binet-like formula for $a_n$. In this particular case, we find $a_n = 3 \cdot 2^n - 2(-1)^n$.