Stability of autonomous linear differential equations

Example 82. (spiral source, spiral sink, center point)

(a) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(c) Analyze the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Solution.

(a)

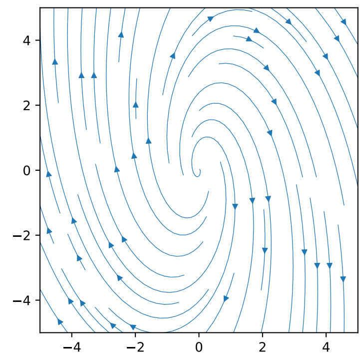
The eigenvalues are $\lambda = 1 \pm 2i$ and the general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}e^t + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}e^t$$

In this case, the origin is a **spiral source** which is an unstable equilibrium (note that it follows from $e^t \to \infty$ as $t \to \infty$ that all solutions "flow away" from the origin because they have increasing amplitude).

Review. $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ parametrizes the unit circle.

Similarly, $\begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$ parametrizes an ellipse.



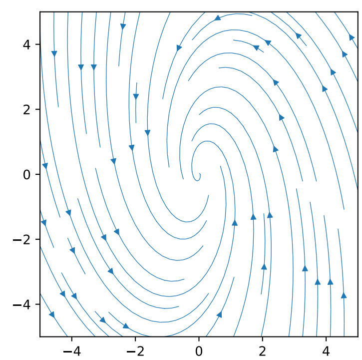
(b)

The eigenvalues are $\lambda = -1 \pm 2i$ and the general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}e^{-t} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}e^{-t}$$

In this case, the origin is a **spiral sink** which is an asymptotically stable equilibrium (note that it follows from $e^{-t} \to 0$ as $t \to \infty$ that all solutions "flow into" the origin because their amplitude goes to zero).

Comment. Note that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ solves the first system if and only if $\begin{bmatrix} x(-t) \\ y(-t) \end{bmatrix}$ is a solution to the second. Consequently, the phase portraits look alike but all arrows are reversed.

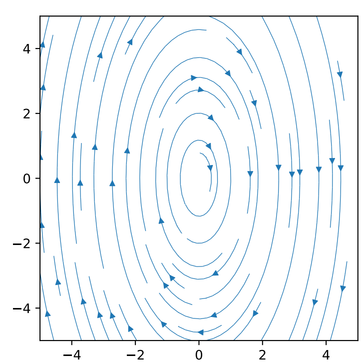


(c)

The eigenvalues are $\lambda = \pm 2i$ and the general solution, in real form, is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1\begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + C_2\begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$

In this case, the origin is a **center point** which is a stable equilibrium (note that the solutions are periodic with period π and therefore loop around the origin; with each trajectory a perfect ellipse).



Review. In Example 79, we considered the system $\frac{dx}{dt} = y - 5x$, $\frac{dy}{dt} = 4x - 2y$.

We found that it has general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

In particular, the only equilibrium point is $(0,0)$ and it is asymptotically stable.

The following example is an inhomogeneous version of Example 79:

Example 83. Analyze the system $\frac{dx}{dt} = y - 5x + 3$, $\frac{dy}{dt} = 4x - 2y$.

In particular, determine the general solution as well as all equilibrium points and their stability.

Solution. As reviewed above, we looked at the corresponding homogeneous system in Example 79 and found that its general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

Note that we can write the present system in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$.

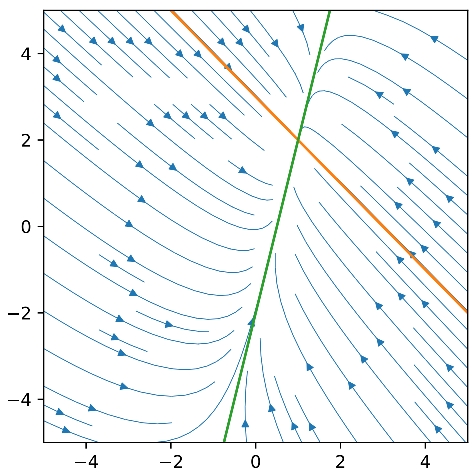
To find the equilibrium point, we solve $M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 0$ to find $\begin{bmatrix} x \\ y \end{bmatrix} = -M^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The fact that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an equilibrium point means that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a particular solution!

(Make sure that you see that it has exactly the form we expect from the method of undetermined coefficients!)

Thus, the general solution must be $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$ (that is, the particular solution plus the general solution of the homogeneous system that we solved in Example 79).

As a result, the phase portrait is going to look just as in Example 79 but shifted by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$:



Because both eigenvalues (-1 and -6) are negative, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an asymptotically stable equilibrium point. More precisely, it is what is called a **nodal source**.

Armin Straub
straub@southalabama.edu

As we have started to observe, the eigenvalues determine the stability of the equilibrium point in the case of an autonomous linear 2-dimensional systems. The following table gives an overview.

Important. Note that such a system must be of the form $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a constant vector. Because the system is autonomous, the matrix M and the inhomogeneous part $\boldsymbol{c}$ cannot depend on t.

(stability of autonomous linear 2-dimensional systems)

eigenvalues	**behaviour**	**stability**	**solutions have terms like**
real and both positive	nodal source	unstable	e^{3t}, e^{7t}
real and both negative	nodal sink	asymptotically stable	e^{-3t}, e^{-7t}
real and opposite signs	saddle	unstable	e^{-3t}, e^{7t}
complex with positive real part	spiral source	unstable	$e^{3t}\cos(7t)$, $e^{3t}\sin(7t)$
complex with negative real part	spiral sink	asymptotically stable	$e^{-3t}\cos(7t)$, $e^{-3t}\sin(7t)$
purely imaginary	center point	stable (not asymptotically stable)	$\cos(7t)$, $\sin(7t)$

Review: Linearizations of nonlinear functions

Recall from Calculus I that a function $f(x)$ around a point x_0 has the linearization

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Here, the right-hand side is the linearization and we also know it as the tangent line to $f(x)$ at x_0.

Recall from Calculus III that a function $f(x, y)$ around a point (x_0, y_0) has the linearization

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Again, the right-hand side is the linearization. This time, it describes the tangent plane to $f(x, y)$ at (x_0, y_0). Recall that $f_x = \frac{\partial}{\partial x} f(x, y)$ and $f_y = \frac{\partial}{\partial y} f(x, y)$ are the partial derivatives of f.

Example 84. Determine the linearization of the function $3 + 2xy^2$ at $(2, 1)$.

Solution. If $f(x, y) = 3 + 2xy^2$, then $f_x = 2y^2$ and $f_y = 4xy$. In particular, $f_x(2, 1) = 2$ and $f_y(2, 1) = 8$. Accordingly, the linearization is $f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 7 + 2(x - 2) + 8(y - 1)$.

Armin Straub
straub@southalabama.edu