

Phase portraits of autonomous linear differential equations

Example 79. Consider the system $\frac{dx}{dt} = y - 5x$, $\frac{dy}{dt} = 4x - 2y$.

- Determine the general solution.
- Make a phase portrait. Can you connect it with the general solution?
- Determine all equilibrium points and their stability.

Solution.

(a) Note that we can write this in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$.

M has -1 -eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as -6 -eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Hence, the general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.

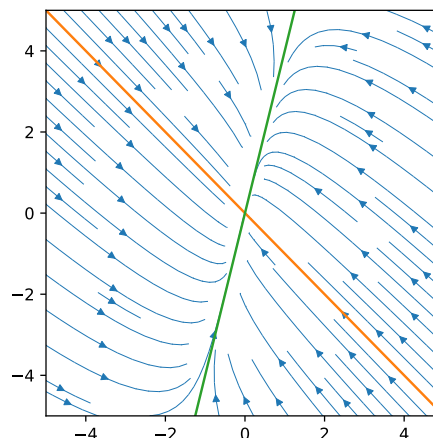
(b) We can have Sage make such a plot for us:

```
>>> x,y = var('x y')
streamline_plot((-5*x+y,4*x-2*y), (x,-4,4), (y,-4,4))
```

Question. In our plot, we also highlighted two lines through the origin. Can you explain their significance?

Explanation. The lines correspond to the special solutions $C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$ (green) and $C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$ (orange). For each, the trajectories consist of points that are multiples of the vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively.

Note that each such solution starts at a point on one of the lines and then “flows” into the origin. (Because e^{-t} and e^{-6t} approach zero for large t .)



Question. Consider a point like $(4, 4)$. Can you explain why the trajectory through that point doesn't go somewhat straight to $(0, 0)$ but rather flows nearly parallel to the orange line towards the green line?

Explanation. A solution through $(4, 4)$ is of the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$ (like any other solution). Note that, if we increase t , then e^{-6t} becomes small much faster than e^{-t} .

As a consequence, we quickly get $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$, where the right-hand side is on the green line.

(c) The only equilibrium point is $(0, 0)$ and it is asymptotically stable.

We can see this from the phase portrait but we can also determine it from the DE and our solution: first, solving $y - 5x = 0$ and $4x - 2y = 0$ we only get the unique solution $x = 0$, $y = 0$, which means that only $(0, 0)$ is an equilibrium point. On the other hand, the general solution shows that every solution approaches $(0, 0)$ as $t \rightarrow \infty$ because both e^{-t} and e^{-6t} approach 0.

In general. This is typical: if both eigenvalues are negative, then the equilibrium is asymptotically stable. If at least one eigenvalue is positive, then the equilibrium is unstable.

Example 80. Consider the system $\frac{dx}{dt} = 5x - y$, $\frac{dy}{dt} = 2y - 4x$.

- Determine the general solution.
- Make a phase portrait.
- Determine all equilibrium points and their stability.

Solution.

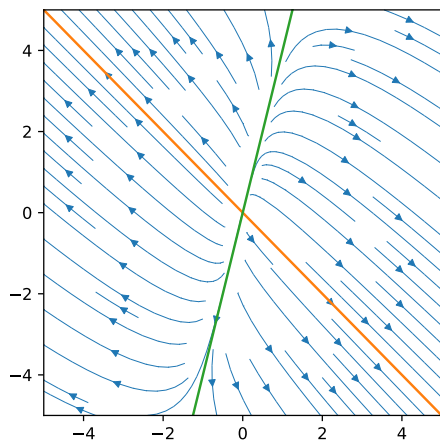
- Note that we can write this in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = -\begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$, where the matrix is exactly -1 times what it was in Example 79.

Consequently, M has 1-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as 6-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (Can you explain why the eigenvectors are the same and the eigenvalues changed sign?)

Thus, the general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$.

- We again have Sage make the plot for us:

```
>>> x,y = var('x y')
streamline_plot((5*x-y,-4*x+2*y), (x,-4,4), (y,-4,4))
```



Note that the phase portrait is identical to the one in Example 79, except that the arrows are reversed.

- The only equilibrium point is $(0,0)$ and it is unstable.

We can see this from the phase portrait but we can also see it readily from our general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ because e^t and e^{6t} go to ∞ as $t \rightarrow \infty$.

In general. If at least one eigenvalue is positive, then the equilibrium is unstable.

Example 81. Suppose the system $\frac{dx}{dt} = f(x, y)$, $\frac{dy}{dt} = g(x, y)$ has general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$. Determine all equilibrium points and their stability.

Solution. Recall that equilibrium points correspond to constant solutions. Clearly, the only constant solution is the zero solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Equivalently, the only equilibrium point is $(0,0)$.

Since $e^{6t} \rightarrow \infty$ as $t \rightarrow \infty$, we conclude that the equilibrium is unstable. (Note that the solution $C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ starts arbitrarily near to $(0,0)$ but always “flows away”).