Phase portraits of autonomous linear differential equations

Example 79. Consider the system $\frac{dx}{dt} = y - 5x$, $\frac{dy}{dt} = 4x - 2y$.

- (a) Determine the general solution.
- (b) Make a phase portrait. Can you connect it with the general solution?
- (c) Determine all equilibrium points and their stability.

Solution.

- (a) Note that we can write this is in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$. M has -1-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as -6-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence, the general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$.
- (b) We can have Sage make such a plot for us:

```
>>> x,y = var('x y')
streamline_plot((-5*x+y,4*x-2*y), (x,-4,4), (y,-4,4))
```

Question. In our plot, we also highlighted two lines through the origin. Can you explain their significance?

Explanation. The lines correspond to the special solutions $C_1 \begin{bmatrix} 1\\4 \end{bmatrix} e^{-t}$ (green) and $C_2 \begin{bmatrix} -1\\1 \end{bmatrix} e^{-6t}$ (orange). For each, the trajectories consist of points that are multiples of the vectors $\begin{bmatrix} 1\\4 \end{bmatrix}$ and $\begin{bmatrix} -1\\1 \end{bmatrix}$, respectively.

Note that each such solution starts at a point on one of the lines and then "flows" into the origin. (Because e^{-t} and e^{-6t} approach zero for large t.)



Question. Consider a point like (4, 4). Can you explain why the trajectory through that point doesn't go somewhat straight to (0, 0) but rather flows nearly parallel the orange line towards the green line? Explanation. A solution through (4, 4) is of the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-6t}$ (like any other solution). Note that, if we increase t, then e^{-6t} becomes small much faster than e^{-t} . As a consequence, we quickly get $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}$, where the right-hand side is on the green line.

(c) The only equilibrium point is (0,0) and it is asymptotically stable.

We can see this from the phase portrait but we can also determine it from the DE and our solution: first, solving y - 5x = 0 and 4x - 2y = 0 we only get the unique solution x = 0, y = 0, which means that only (0,0) is an equilibrium point. On the other hand, the general solution shows that every solution approaches (0,0) as $t \to \infty$ because both e^{-t} and e^{-6t} approach 0.

In general. This is typical: if both eigenvalues are negative, then the equilibrium is asymptotically stable. If at least one eigenvalue is positive, then the equilibrium is unstable.

Example 80. Consider the system $\frac{dx}{dt} = 5x - y$, $\frac{dy}{dt} = 2y - 4x$.

- (a) Determine the general solution.
- (b) Make a phase portrait.
- (c) Determine all equilibrium points and their stability.

Solution.

- (a) Note that we can write this is in matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}' = M \begin{bmatrix} x \\ y \end{bmatrix}$ with $M = -\begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix}$, where the matrix is exactly -1 times what it was in Example 79. Consequently, M has 1-eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as well as 6-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (Can you explain why the eigenvectors are the same and the eigenvalues changed sign?) Thus, the general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$.
- (b) We again have Sage make the plot for us:

```
>>> x,y = var('x y')
streamline_plot((5*x-y,-4*x+2*y), (x,-4,4), (y,-4,4))
```



Note that the phase portrait is identical to the one in Example 79, except that the arrows are reversed.

(c) The only equilibrium point is (0,0) and it is unstable.

We can see this from the phase portrait but we can also see it readily from our general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ because e^t and e^{6t} go to ∞ as $t \to \infty$.

In general. If at least one eigenvalue is positive, then the equilibrium is unstable.

Example 81. Suppose the system $\frac{dx}{dt} = f(x, y)$, $\frac{dy}{dt} = g(x, y)$ has general solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$. Determine all equilibrium points and their stability.

Solution. Recall that equilibrium points correspond to constant solutions. Clearly, the only constant solution is the zero solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Equivalently, the only equilibrium point is (0,0). Since $e^{6t} \to \infty$ as $t \to \infty$, we conclude that the equilibrium is unstable. (Note that the solution $C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t}$ starts arbitrarily near to (0,0) but always "flows away").