**Example 71.** Suppose that $e^{Mx} = \frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix}$.

(a) Without doing any computations, determine $M^n$.

(b) What is $M$?

(c) Without doing any computations, determine the eigenvalues and eigenvectors of $M$.

(d) From those, write down a simple fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(e) From that fundamental matrix solution, how can we compute $e^{Mx}$? (If we didn't know it already...)

(f) Having computed $e^{Mx}$, what is a simple check that we can (should!) make?

**Solution.**

(a) Since $e^x$ and $e^{2x}$ correspond to eigenvalues $1$ and $2$, we just need to replace these by $1^n = 1$ and $2^n$:

$$M^n = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2^n & 3 - 3 \cdot 2^n \\ 3 - 3 \cdot 2^n & 9 + 2^n \end{bmatrix}$$

(b) We can simply set $n = 1$ in our formula for $M^n$, to get $M = \frac{1}{10}\begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$.

(c) The eigenvalues are $1$ and $2$ (because $e^{Mx}$ contains the exponentials $e^x$ and $e^{2x}$).
Looking at the coefficients of $e^x$ in the first column of $e^{Mx}$, we see that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a $1$-eigenvector.
[We can also look the second column of $e^{Mx}$, to obtain $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ which is a multiple and thus equivalent.]
Likewise, by looking at the coefficients of $e^{2x}$, we see that $\begin{bmatrix} 9 \\ -3 \end{bmatrix}$ or, equivalently, $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a $2$-eigenvector.

**Comment.** To see where this is coming from, keep in mind that, associated to a $\lambda$-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ of the DE $\boldsymbol{y}' = M\boldsymbol{y}$. On the other hand, the columns of $e^{Mx}$ are solutions to that DE and, therefore, must be linear combinations of these $\boldsymbol{v}e^{\lambda x}$.

(d) From the eigenvalues and eigenvectors, we know that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^x$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{2x}$ are solutions (and that the general solutions consists of the linear combinations of these two).
Selecting these as the columns, we obtain the fundamental matrix solution $\Phi(x) = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}$.

**Comment.** The *fundamental* refers to the fact that the columns combine to the general solution.
The *matrix solution* means that $\Phi(x)$ itself satisfies the DE: namely, we have $\Phi' = M\Phi$. That this is the case is a consequence of matrix multiplication (namely, say, the second column of $M\Phi$ is defined to be $M$ times the second column of $\Phi$; but that column is a vector solution and therefore solves the DE).

(e) We can compute $e^{Mx}$ as $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.
If $\Phi(x) = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}$, then $\Phi(0) = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and, hence, $\Phi(0)^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} e^x & -3e^{2x} \\ 3e^x & e^{2x} \end{bmatrix}\frac{1}{10}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix}.$$

(f) We can check that $e^{Mx}$ equals the identity matrix if we set $x = 0$:

$$\frac{1}{10}\begin{bmatrix} e^x + 9e^{2x} & 3e^x - 3e^{2x} \\ 3e^x - 3e^{2x} & 9e^x + e^{2x} \end{bmatrix} \overset{x=0}{\rightsquigarrow} \frac{1}{10}\begin{bmatrix} 1 + 9 & 3 - 3 \\ 3 - 3 & 9 + 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This check does not require much effort and can even be done in our head while writing down $e^{Mx}$. There is really no excuse for not doing it!

## Another perspective on the matrix exponential

**Review.** We achieved the milestone to introduce a **matrix exponential** in such a way that we can treat a system of DEs, say $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \boldsymbol{c}$, just as if the matrix $M$ was a number: namely, the unique solution is simply $\boldsymbol{y} = e^{Mx}\boldsymbol{c}$.

The price to pay is that the matrix $e^{Mx}$ requires some work to actually compute (and proceeds by first determining a different matrix solution $\Phi(x)$ using eigenvectors and eigenvalues). We offer below another way to think about $e^{Mx}$ (using Taylor series).

**(exponential function)** $e^x$ is the unique solution to $y' = y$, $y(0) = 1$.

From here, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

The latter is the Taylor series for $e^x$ at $x = 0$ that we have seen in Calculus II.

**Important note.** We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.

Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!} = \frac{x^2}{2!}$.

**Review.** We defined the **matrix exponential** $e^{Mx}$ as the unique matrix solution to the IVP

$$\boldsymbol{y}' = M\boldsymbol{y}, \quad \boldsymbol{y}(0) = I.$$

We next observe that we can also make sense of the matrix exponential $e^{Mx}$ as a power series.

**Theorem 72.** Let $M$ be $n \times n$. Then the **matrix exponential** satisfies

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \ldots$$

**Proof.** Define $\Phi(x) = I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots$

$$\begin{aligned}\Phi'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots\right]\\ &= 0 + M + M^2x + \frac{1}{2!}M^3x^2 + \ldots = M\Phi(x).\end{aligned}$$

Clearly, $\Phi(0) = I$. Therefore, $\Phi(x)$ is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$.

But that's precisely how we defined $e^{Mx}$ earlier. It follows that $\Phi(x) = e^{Mx}$. Now set $x = 1$. □

**Example 73.** If $A = \begin{bmatrix} 2 & 0\\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0\\ 0 & 5^{100} \end{bmatrix}$.

**Example 74.** If $A = \begin{bmatrix} 2 & 0\\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0\\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0\\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0\\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain $e^D$ for every diagonal matrix $D$.

In particular, for $Ax = \begin{bmatrix} 2x & 0\\ 0 & 5x \end{bmatrix}$, $e^{Ax} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2x & 0\\ 0 & 5x \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2x)^2 & 0\\ 0 & (5x)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2x} & 0\\ 0 & e^{5x} \end{bmatrix}$.

The following is a preview of how the matrix exponential deals with repeated characteristic roots.

**Example 75.** Determine $e^{Ax}$ for $A = \begin{bmatrix} 0 & 1\\ 0 & 0 \end{bmatrix}$.

**Solution.** If we compute eigenvalues, we find that we get $\lambda = 0, 0$ (multiplicity $2$) but there is only one $0$-eigenvector (up to multiples). This means we are stuck with this approach—however, see next extra section how we could still proceed.

The key here is to observe that $A^2 = \begin{bmatrix} 0 & 0\\ 0 & 0 \end{bmatrix}$. It follows that $e^{Ax} = I + Ax = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & x\\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x\\ 0 & 1 \end{bmatrix}$.

## Extra: The case of repeated eigenvalues with too few eigenvectors

**Review.** To construct a fundamental matrix solution $\Phi(x)$ to $\boldsymbol{y}' = M\boldsymbol{y}$, we compute eigenvectors:
Given a $\lambda$-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
If there are enough eigenvectors, we can collect these as columns to obtain $\Phi(x)$.
The next example illustrates how to proceed if there are not enough eigenvectors.

In that case, instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. This can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

**Example 76.** Let $M = \begin{bmatrix} 8 & 4 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute $e^{Mx}$.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

**Solution.**

(a) We determine the eigenvectors of $M$. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix}\right) = (8-\lambda)(4-\lambda) + 4 = \lambda^2 - 12\lambda + 36 = (\lambda - 6)(\lambda - 6)$

Hence, the eigenvalues are $\lambda = 6, 6$ (meaning that $6$ has multiplicity $2$).

- To find eigenvectors $\boldsymbol{v}$ for $\lambda = 6$, we need to solve $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.
  Hence, $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 6$. There is no independent second eigenvector.

- We therefore search for a solution of the form $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$ with $\lambda = 6$.
  $\boldsymbol{y}'(x) = (\lambda\boldsymbol{v}x + \lambda\boldsymbol{w} + \boldsymbol{v})e^{\lambda x} \overset{!}{=} M\boldsymbol{y} = (M\boldsymbol{v}x + M\boldsymbol{w})e^{\lambda x}$
  Equating coefficients of $x$, we need $\lambda\boldsymbol{v} = M\boldsymbol{v}$ and $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$.
  Hence, $\boldsymbol{v}$ must be an eigenvector (which we already computed); we choose $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
  [Note that any multiple of $\boldsymbol{y}(x)$ will be another solution, so it doesn't matter which multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ we choose.]
  $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$ or $(M - \lambda)\boldsymbol{w} = \boldsymbol{v}$ then becomes $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
  One solution is $\boldsymbol{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. [We only need one.]

Hence, the general solution is $C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{6x} + C_2\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}x + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)e^{6x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & xe^{6x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & xe^{6x} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4xe^{6x} \\ -xe^{6x} & -(2x-1)e^{6x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4xe^{6x} \\ -xe^{6x} & -(2x-1)e^{6x} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} \\ -xe^{6x} \end{bmatrix}$.