## Solving systems of recurrence equations

The following summarizes how we can solve systems of recurrence equations using eigenvectors. As a bonus, we obtain a way to compute matrix powers.

Each step is spelled out in Example 60 below.

**(solving systems of REs)** To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, determine the eigenvectors of $M$.

- Each $\lambda$-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ [assuming that $\lambda \neq 0$]
- If there are enough eigenvectors, these combine to the general solution.
  In that case, we get a **fundamental matrix (solution)** $\Phi_n$ by placing each solution vector into one column of $\Phi_n$.
- If desired, we can compute the **matrix powers** $M^n$ using any fundamental matrix $\Phi_n$ as

$$M^n = \Phi_n \Phi_0^{-1}.$$

Note that $M^n$ is the unique matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = I$ (the identity matrix).

Application: the unique solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \boldsymbol{c}$ is given by $\boldsymbol{a}_n = M^n\boldsymbol{c}$.

**Why?** If $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ for a $\lambda$-eigenvector $\boldsymbol{v}$, then $\boldsymbol{a}_{n+1} = \boldsymbol{v}\lambda^{n+1}$ and $M\boldsymbol{a}_n = M\boldsymbol{v}\lambda^n = \lambda\boldsymbol{v}\cdot\lambda^n = \boldsymbol{v}\lambda^{n+1}$.

**Where is this coming from?** When solving single linear recurrences, we found that the basic solutions are of the form $cr^n$ where $r \neq 0$ is a root of the characteristic polynomials. To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, it is therefore natural to look for solutions of the form $\boldsymbol{a}_n = \boldsymbol{c}r^n$ (where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$). Note that $\boldsymbol{a}_{n+1} = \boldsymbol{c}r^{n+1} = r\boldsymbol{a}_n$.

Plugging into $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ we find $\boldsymbol{c}r^{n+1} = M\boldsymbol{c}r^n$.
Cancelling $r^n$ (just a nonzero number!), this simplifies to $r\boldsymbol{c} = M\boldsymbol{c}$.
In other words, $\boldsymbol{a}_n = \boldsymbol{c}r^n$ is a solution if and only if $\boldsymbol{c}$ is an $r$-eigenvector of $M$.

**Not enough eigenvectors?** In that case, we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$, we also need to look for solutions of the type $\boldsymbol{a}_n = (\boldsymbol{v}n + \boldsymbol{w})\lambda^n$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

**Matrix solutions.** A matrix $\Phi_n$ is a **matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ if $\Phi_{n+1} = M\Phi_n$. $\Phi_n$ being a matrix solution is equivalent to each column of $\Phi_n$ being a normal (vector) solution. If the general solution of $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ can be obtained as the linear combination of the columns of $\Phi_n$, then $\Phi_n$ is a **fundamental matrix solution**.

**Why can we compute matrix powers this way?** Recall that, given a first-order system $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, it is clear that the solution satisfies $\boldsymbol{a}_n = M^n\boldsymbol{a}_0$. Likewise, a fundamental matrix solution $\Phi_n$ to the same recurrence satisfies $\Phi_n = M^n\Phi_0$. Multiplying both sides by $\Phi_0^{-1}$ (on the right!) we conclude that $\Phi_n\Phi_0^{-1} = M^n$.

**Already know how to compute matrix powers?** If you have taken linear algebra classes, you may have learned that matrix powers $M^n$ can be computed by diagonalizing the matrix $M$. The latter hinges on computing eigenvalues and eigenvectors of $M$ as well. Compare the two approaches!

**Example 60.** Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a **fundamental matrix solution** to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute $M^n$.

(d) Solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.


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**Solution.**

(a) Recall that each $\lambda$-eigenvector $\boldsymbol{v}$ of $M$ provides us with a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$

We computed in Example 54 that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Hence, the general solution is $C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} 3^n + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-2)^n$.

(b) Note that we can write the general solution as

$\boldsymbol{a}_n = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} 3^n + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-2)^n = \begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

We call $\Phi_n = \begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}$ the corresponding **fundamental matrix (solution)**.

Note that our general solution is precisely $\Phi_n \boldsymbol{c}$ with $\boldsymbol{c} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

**Observations.**

(a) The columns of $\Phi_n$ are (independent) solutions of the system.

(b) $\Phi_n$ solves the RE itself: $\Phi_{n+1} = M\Phi_n$.
[Spell this out in this example! That $\Phi_n$ solves the RE follows from the definition of matrix multiplication.]

(c) It follows that $\Phi_n = M^n \Phi_0$. Equivalently, $\Phi_n \Phi_0^{-1} = M^n$. (See next part!)

(c) Note that $\Phi_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3^n - (-2)^n & -2 \cdot 3^n + 2(-2)^n \\ 3^n - (-2)^n & -3^n + 2(-2)^n \end{bmatrix}.$$

**Check.** Let us verify the formula for $M^n$ in the cases $n = 0$ and $n = 1$:

$M^0 = \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M^1 = \begin{bmatrix} 2 \cdot 3 - (-2) & -2 \cdot 3 + 2(-2) \\ 3 - (-2) & -3 + 2(-2) \end{bmatrix} = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

(d) $\boldsymbol{a}_n = M^n \boldsymbol{a}_0 = \begin{bmatrix} 2 \cdot 3^n - (-2)^n & -2 \cdot 3^n + 2(-2)^n \\ 3^n - (-2)^n & -3^n + 2(-2)^n \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \cdot 3^n + 3(-2)^n \\ -3^n + 3(-2)^n \end{bmatrix}$

**Sage.** Once we are comfortable with these computations, we can let Sage do them for us.

```
>>> M = matrix([[8,-10],[5,-7]])
```

```
>>> M^2
```

$$\begin{pmatrix} 14 & -10 \\ 5 & -1 \end{pmatrix}$$

Verify that this matrix matches what our formula for $M^n$ produces for $n = 2$. In order to reproduce the general formula for $M^n$, we need to first define $n$ as a symbolic variable:

```
>>> n = var('n')
```

```
>>> M^n
```

$$\begin{pmatrix} 2 \cdot 3^n - (-2)^n & -2 \cdot 3^n + 2\,(-2)^n \\ 3^n - (-2)^n & -3^n + 2\,(-2)^n \end{pmatrix}$$

Note that this indeed matches our earlier formula. Can you see how we can read off the eigenvalues and eigenvectors of $M$ from this formula for $M^n$? Of course, Sage can readily compute these for us directly using, for instance, `M.eigenvectors_right()`. Try it! Can you interpret the output?

**Example 61. (review)** Write the (second-order) RE $a_{n+2} = a_{n+1} + 2a_n$, with $a_0 = 0$, $a_1 = 1$, as a system of (first-order) recurrences.

**Solution.** If $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, then $\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+1} + 2a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**Example 62.** Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute $M^n$.

(d) Solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**Solution.**

(a) Recall that each $\lambda$-eigenvector $\boldsymbol{v}$ of $M$ provides us with a solution: namely, $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$.

The characteristic polynomial is: $\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.

Hence, the eigenvalues are $\lambda = 2$ and $\lambda = -1$.

- $\lambda = 2$: Solving $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.
- $\lambda = -1$: Solving $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$, we find that $\boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

Hence, the general solution is $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2^n + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (-1)^n$.

(b) Note that $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2^n + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (-1)^n = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Hence, a fundamental matrix solution is $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix}$.

**Comment.** Other choices are possible and natural. For instance, the order of the two columns is based on our choice of starting with $\lambda = 2$. Also, the columns can be scaled by any constant (for instance, using $-\boldsymbol{v}$ instead of $\boldsymbol{v}$ for $\lambda = -1$ above, we end up with the same $\Phi_n$ but with the second column scaled by $-1$). In general, if $\Phi_n$ is a fundamental matrix solution, then so is $\Phi_n C$ where $C$ is an invertible $2 \times 2$ matrix.

(c) We compute $M^n = \Phi_n \Phi_0^{-1}$ using $\Phi_n = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix}$. Since $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, we have

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 2^n & -(-1)^n \\ 2 \cdot 2^n & (-1)^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2 \cdot 2^n - 2(-1)^n & 2 \cdot 2^n + (-1)^n \end{bmatrix}.$$

(d) $\boldsymbol{a}_n = M^n \boldsymbol{a}_0 = \frac{1}{3} \begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2 \cdot 2^n - 2(-1)^n & 2 \cdot 2^n + (-1)^n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^n - (-1)^n \\ 2 \cdot 2^n + (-1)^n \end{bmatrix}$

**Alternative solution of the first part.** We saw in Example 61 that this system can be obtained from $a_{n+2} = a_{n+1} + 2a_n$ if we set $\boldsymbol{a} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. In Example 46, we found that this RE has solutions $a_n = 2^n$ and $a_n = (-1)^n$.

Correspondingly, $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{a}_n$ has solutions $\boldsymbol{a}_n = \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix}$ and $\boldsymbol{a}_n = \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$.

These combine to the general solution $C_1 \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix} + C_2 \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$ (equivalent to our solution above).

**Alternative for last part.** Solve the RE from Example 61 to find $a_n = \frac{1}{3}(2^n - (-1)^n)$. The above is $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$.