Crash course: Eigenvalues and eigenvectors

If $Ax = \lambda x$ (and $x \neq 0$), then x is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that, for the equation $Ax = \lambda x$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$Ax = \lambda x$$
$$\Longleftrightarrow \quad Ax - \lambda x = 0$$
$$\Longleftrightarrow \quad (A - \lambda I)x = 0$$

This homogeneous system has a nontrivial solution x if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

   (a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

   $\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

   (b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)x = 0$.

Example 54. Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

Solution. The characteristic polynomial is:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{bmatrix} \right) = (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$$

Hence, the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

- To find an eigenvector for $\lambda = 3$, we need to solve $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} x = 0$.

  Hence, $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.

- To find an eigenvector for $\lambda = -2$, we need to solve $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} x = 0$.

  Hence, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Check! $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

On the other hand, a random other vector like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector: $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Example 55. (homework) Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix}$.

Solution. (final answer only) $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$, and $x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

Example 56. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits pro-
duces one pair of baby rabbit as offspring.
Meanwhile, it takes baby rabbits one month
to mature to adults.
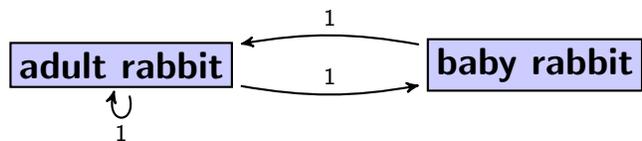
Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these features might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

If we start with one baby rabbit pair, how many adult rabbit pairs are there after n months?

Solution. Let a_n be the number of adult rabbit pairs after n months. Likewise, b_n is the number of baby rabbit pairs. The transition from one month to the next is given by $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$. Using $b_n = a_{n-1}$ (from the second equation) in the first equation, we obtain $a_{n+1} = a_n + a_{n-1}$.

The initial conditions are $a_0 = 0$ and $a_1 = 1$ (the latter follows from $b_0 = 1$).

It follows that the number b_n of adult rabbit pairs are precisely the Fibonacci numbers F_n.

Comment. Note that the transition from one month to the next is described by in matrix-vector form as

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} a_n + b_n \\ a_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

Writing $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$, this becomes $\boldsymbol{a}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n$ with $\boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Consequently, $\boldsymbol{a}_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Looking ahead. Can you see how, starting with the Fibonacci recurrence $F_{n+2} = F_{n+1} + F_n$, we can arrive at this same system?

Solution. Set $\boldsymbol{a}_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$. Then $\boldsymbol{a}_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n$.

Systems of recurrence equations

Example 57. (review) Consider the sequence a_n defined by $a_{n+2} = 4a_n - 3a_{n+1}$ and $a_0 = 1$, $a_1 = 2$. Determine $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 + 3N - 4$ has roots $1, -4$. Hence, the general solution is $a_n = C_1 + C_2 \cdot (-4)^n$. We can see that both roots have to be involved in the solution (in other words, $C_1 \neq 0$ and $C_2 \neq 0$) because $a_n = C_1$ and $a_n = C_2 \cdot (-4)^n$ are not consistent with the initial conditions. We conclude that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = -4$ (because $|-4| > |1|$).

Example 58. Write the (second-order) RE $a_{n+2} = 4a_n - 3a_{n+1}$, with $a_0 = 1$, $a_1 = 2$, as a system of (first-order) recurrences.

Solution. Write $b_n = a_{n+1}$.

Then, $a_{n+2} = 4a_n - 3a_{n+1}$ translates into the first-order system $\begin{cases} a_{n+1} = b_n \\ b_{n+1} = 4a_n - 3b_n \end{cases}$.

Let $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$. Then, in matrix form, the RE is $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \boldsymbol{a}_n$, with $\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Equivalently. Write $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. Then we obtain the above system as

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ 4a_n - 3a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \boldsymbol{a}_n, \quad \boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Comment. It follows that $\boldsymbol{a}_n = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Solving (systems of) REs is equivalent to computing powers of matrices!

Comment. We could also write $\boldsymbol{a}_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ (with the order of the entries reversed). In that case, the system is

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4a_n - 3a_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix} \boldsymbol{a}_n, \quad \boldsymbol{a}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Comment. Recall that the **characteristic polynomial** of a matrix M is $\det(M - \lambda I)$. Compute the characteristic polynomial of both $M = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}$ and $M = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}$. In both cases, we get $\lambda^2 + 3\lambda - 4$, which matches the polynomial $p(N)$ (also called characteristic polynomial!) in the previous example. This will always happen and explains why both are referred to as the characteristic polynomial.

Example 59. Write $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ as a system of (first-order) recurrences.

Solution. Write $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$. Then we obtain the system

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ 4a_{n+2} - a_{n+1} - 6a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \boldsymbol{a}_n.$$

In summary, the RE in matrix form is $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with M the matrix above.

Important comment. Given a first-order system $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, it is clear that the solution satisfies $\boldsymbol{a}_n = M^n \boldsymbol{a}_0$. If you know how to compute matrix powers M^n, this means you can solve recurrences! On the other hand, we will proceed the other way around. We solve the recurrence and then use that to determine M^n.