

Review. The recurrence $a_{n+1} = 5a_n$ has general solution $a_n = C \cdot 5^n$.

In operator form, the recurrence is $(N - 5)a_n = 0$, where $p(N) = N - 5$ is the characteristic polynomial. The characteristic root 5 corresponds to the solution 5^n .

This is analogous to the case of DEs $p(D)y = 0$ where a root r of $p(D)$ corresponds to the solution e^{rx} .

Example 45. (cont'd) Let the sequence a_n be defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 1, a_1 = 8$.

- Determine the first few terms of the sequence.
- Find a formula for a_n .
- Determine $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = a_1 + 6a_0 = 14, a_3 = a_2 + 6a_1 = 62, a_4 = 146, \dots$

- (b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6$ has roots 3, -2.

Hence, $a_n = C_1 3^n + C_2 (-2)^n$ and we only need to figure out the two unknowns C_1, C_2 . We can do that using the two initial conditions: $a_0 = C_1 + C_2 = 1, a_1 = 3C_1 - 2C_2 = 8$.

Solving, we find $C_1 = 2$ and $C_2 = -1$ so that, in conclusion, $a_n = 2 \cdot 3^n - (-2)^n$.

Comment. Such a formula is sometimes called a **Binet-like formula** (because it is of the same kind as the Binet formula for the Fibonacci numbers that we can derive in the same manner).

- (c) It follows from our formula that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 3$ (because $|3| > |-2|$ so that 3^n dominates $(-2)^n$).

To see this, we need to realize that, for large n , 3^n is much larger than $(-2)^n$ so that we have $a_n \approx 2 \cdot 3^n$ when n is large. Hence, $\frac{a_{n+1}}{a_n} \approx \frac{2 \cdot 3^{n+1}}{2 \cdot 3^n} = 3$.

Alternatively, to be very precise, we can observe that (by dividing each term by 3^n)

$$\frac{a_{n+1}}{a_n} = \frac{2 \cdot 3^{n+1} - (-2)^{n+1}}{2 \cdot 3^n - (-2)^n} = \frac{2 \cdot 3 + 2 \left(-\frac{2}{3} \right)^n}{2 \cdot 1 - \left(-\frac{2}{3} \right)^n} \quad \text{as } n \rightarrow \infty \quad \frac{2 \cdot 3 + 0}{2 \cdot 1 - 0} = 3.$$

Example 46. Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 2a_n$ and $a_0 = 1, a_1 = 8$.

- Determine the first few terms of the sequence.
- Find a formula for a_n .
- Determine $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = 10, a_3 = 26$

- (b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 2$ has roots 2, -1.

Hence, $a_n = C_1 2^n + C_2 (-1)^n$ and we only need to figure out the two unknowns C_1, C_2 . We can do that using the two initial conditions: $a_0 = C_1 + C_2 = 1, a_1 = 2C_1 - C_2 = 8$.

Solving, we find $C_1 = 3$ and $C_2 = -2$ so that, in conclusion, $a_n = 3 \cdot 2^n - 2(-1)^n$.

- (c) It follows from the formula $a_n = 3 \cdot 2^n - 2(-1)^n$ that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2$.

Comment. In fact, this already follows from $a_n = C_1 2^n + C_2 (-1)^n$ provided that $C_1 \neq 0$. Since $a_n = C_2 (-1)^n$ (the case $C_1 = 0$) is not compatible with $a_0 = 1, a_1 = 8$, we can conclude $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2$ without computing the actual values of C_1 and C_2 .

Example 47. (“warmup”) Find the general solution to the recursion $a_{n+2} = 4a_{n+1} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N + 4$ has roots 2, 2.

So a solution is 2^n and, from our discussion of DEs, it is probably not surprising that a second solution is $n \cdot 2^n$.

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n = (C_1 + C_2 n) \cdot 2^n$.

Comment. This is analogous to $(D - 2)^2 y' = 0$ having the general solution $y(x) = (C_1 + C_2 x)e^{2x}$.

Check! Let’s check that $a_n = n \cdot 2^n$ indeed satisfies the recursion $(N - 2)^2 a_n = 0$.

$(N - 2)n \cdot 2^n = (n + 1)2^{n+1} - 2n \cdot 2^n = 2^{n+1}$, so that $(N - 2)^2 n \cdot 2^n = (N - 2)2^{n+1} = 0$.

Combined, we obtain the following analog of Theorem 20 for recurrence equations (RE):

Comment. Sequences that are solutions to such recurrences are called **constant recursive** or **C-finite**.

Theorem 48. Consider the homogeneous linear RE with constant coefficients $p(N)a_n = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the RE are given by $n^j r^n$ for $j = 0, 1, \dots, k - 1$.
- Combining these solutions for all roots, gives the general solution.

Moreover. If r is the sole largest root by absolute value among the roots contributing to a_n , then $a_n \approx Cr^n$ (if r is not repeated—what if it is?) for large n . In particular, it follows that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r.$$

Advanced comment. Things can get weird if there are several roots of the same absolute value. Consider, for instance, the case $a_n = 2^n + (-2)^n$. Can you see that, in this case, the limit $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ doesn’t even exist?

Example 49. Find the general solution to the recursion $a_{n+3} = 2a_{n+2} + a_{n+1} - 2a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 2N^2 - N + 2$ has roots 2, 1, -1. (Here, we may use some help from a computer algebra system to find the roots.)

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 + C_3 \cdot (-1)^n$.

Example 50. Find the general solution to the recursion $a_{n+3} = 3a_{n+2} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 3N^2 + 4$ has roots 2, 2, -1. (Again, we may use some help from a computer algebra system to find the roots.)

Hence, the general solution is $a_n = (C_1 + C_2 n) \cdot 2^n + C_3 \cdot (-1)^n$.

Theorem 51. (Binet's formula) $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$

Proof. The recursion $F_{n+1} = F_n + F_{n-1}$ can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 1$ has roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Hence, $F_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n$ and we only need to figure out the two unknowns C_1, C_2 . We can do that using the two initial conditions: $F_0 = C_1 + C_2 \stackrel{!}{=} 0$, $F_1 = C_1 \cdot \frac{1+\sqrt{5}}{2} + C_2 \cdot \frac{1-\sqrt{5}}{2} \stackrel{!}{=} 1$.

Solving, we find $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$ so that, in conclusion, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, as claimed. \square

Comment. For large n , $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round} \left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n , it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1} \right)^n} \xrightarrow{n \rightarrow \infty} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

In fact, it follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Example 52. Consider the sequence a_n defined by $a_{n+2} = 4a_{n+1} + 9a_n$ and $a_0 = 1, a_1 = 2$. Determine $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N - 9$ has roots $\frac{4 \pm \sqrt{52}}{2} \approx 5.6056, -1.6056$. Both roots have to be involved in the solution in order to get integer values.

We conclude that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2 + \sqrt{13} \approx 5.6056$ (because $|5.6056| > |-1.6056|$).

Example 53. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0, a_1 = 1$. Determine $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

First few terms of sequence. 0, 1, 2, 8, 24, 80, 256, 832, ...

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1}F_n$. Can you prove this directly from the recursions? Alternatively, this follows from comparing the Binet-like formulas.

Solution. Proceeding as in the previous example, we find $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet-like formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.