

Example 34. (review) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part roots are $3 \pm 2i, \pm i, \pm i$. Hence, there has to be a particular solution of the form

$$y_p = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x) + (C_3 + C_4 x) \cos(x) + (C_5 + C_6 x) \sin(x).$$

Continuing to find a particular solution. To find the values of C_1, \dots, C_6 , we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

Sage

In practice, we are happy to let a machine do tedious computations. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at sagemath.org. Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at cocalc.com from any browser.

[For basic computations, you can also simply use the textbox on our course website.]

Sage is built as a **Python** library, so any Python code is valid. For starters, we will use it as a fancy calculator.

Example 35. To solve the differential equation $y'' + 4y' + 4y = 7e^{-2x}$, as we did in Example 29, we can use the following:

```
>>> x = var('x')
>>> y = function('y')(x)
>>> desolve(diff(y,x,2) + 4*diff(y,x) + 4*y == 7*exp(-2*x), y)
```

$$\frac{7}{2} x^2 e^{(-2x)} + (K_2 x + K_1) e^{(-2x)}$$

This confirms, as we had found, that the general solution is $y(x) = (C_1 + C_2 x + \frac{7}{2} x^2) e^{-2x}$.

Example 36. Similarly, Sage can solve initial value problems such as $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

```
>>> x = var('x')
>>> y = function('y')(x)
>>> desolve(diff(y,x,2) - diff(y,x) - 2*y == 0, y, ics=[0,4,5])
```

$$3 e^{(2x)} + e^{(-x)}$$

This matches the (unique) solution $y(x) = 3e^{2x} + e^{-x}$ that we derived in Example 18.

Higher order. Unfortunately, the command `desolve` currently only works like this for differential equations of first and second order. To likewise solve a third-order differential equation, we can use the function `desolve_laplace` instead. For instance, to solve the IVP $y''' = 3y'' - 4y$ with $y(0) = 1$, $y'(0) = -2$, $y''(0) = 3$, use

```
>>> desolve_laplace(diff(y,x,3) == 3*diff(y,x,2) - 4*y, y, ics=[0,1,-2,3])
```

$$x e^{(2x)} - \frac{2}{3} e^{(2x)} + \frac{5}{3} e^{(-x)}$$

to find that the unique solution is $y(x) = \frac{1}{3}(3x - 2)e^{2x} + \frac{5}{3}e^{-x}$.

More on differential operators

Example 37. We have been factoring differential operators like $D^2 + 4D + 4 = (D + 2)^2$.

Things become much more complicated when the coefficients are not constant!

For instance, the linear DE $y'' + 4y' + 4xy = 0$ can be written as $Ly = 0$ with $L = D^2 + 4D + 4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $x D \neq D x$!!

Indeed, $(x D)f(x) = x f'(x)$ while $(D x)f(x) = \frac{d}{dx}[x f(x)] = f(x) + x f'(x) = (1 + x D)f(x)$.

This computation shows that, in fact, $D x = x D + 1$.

Review. Linear DEs are those that can be written as $Ly = f(x)$ where L is a linear differential operator: namely,

$$L = p_n(x)D^n + p_{n-1}(x)D^{n-1} + \dots + p_1(x)D + p_0(x). \quad (1)$$

Recall that the operators $x D$ and $D x$ are not the same: instead, $D x = x D + 1$.

We say that an operator of the form (1) is in **normal form**.

For instance. $x D$ is in normal form, whereas $D x$ is not in normal form. It follows from the previous example that the normal form of $D x$ is $x D + 1$.

Example 38. Let $a = a(x)$ be some function.

- (a) Write the operator $D a$ in normal form [normal form means as in (1)].
(b) Write the operator $D^2 a$ in normal form.

Solution.

(a) $(D a)f(x) = \frac{d}{dx}[a(x)f(x)] = a'(x)f(x) + a(x)f'(x) = (a' + a D)f(x)$
Hence, $D a = a D + a'$.

(b) $(D^2 a)f(x) = \frac{d^2}{dx^2}[a(x)f(x)] = \frac{d}{dx}[a'(x)f(x) + a(x)f'(x)] = a''(x)f(x) + 2a'(x)f'(x) + a(x)f''(x)$
 $= (a'' + 2a'D + aD^2)f(x)$
Hence, $D^2 a = aD^2 + 2a'D + a''$.