Example 25. (review) Find the general solution of $y''' - 3y' + 2y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D + 2 = (D-1)^2(D+2)$ has roots $1, 1, -2$.
By Theorem 20, the general solution is $y(x) = (C_1 + C_2 x)e^x + C_2 e^{-2x}$.

Example 26. (review) Consider the function $y(x) = 7x - 5x^2 e^{4x}$. Find an operator $p(D)$ such that $p(D)y = 0$.

Comment. This is the same as determining a homogeneous linear DE with constant coefficients solved by $y(x)$.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $0, 0, 4, 4, 4$.
The simplest choice for $p(D)$ thus is $p(D) = D^2(D-4)^3$.

Inhomogeneous linear DEs: The method of undetermined coefficients

The **method of undetermined coefficients** allows us to solve certain inhomogeneous linear DEs $Ly = f(x)$ with constant coefficients..

It works if $f(x)$ is itself a solution of a homogeneous linear DE with constant coefficients (see previous example).

Example 27. Determine the general solution of $y'' + 4y = 12x$.

Solution. The DE is $p(D)y = 12x$ with $p(D) = D^2 + 4$, which has roots $\pm 2i$. Thus, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.
Since $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE to get the **homogeneous** DE $D^2(D^2+4) \cdot y = 0$.
Its general solution is $C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x)$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = C_1 + C_2 x$ (because $C_3\cos(2x) + C_4\sin(2x)$ can be added to any y_p).
It remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.
Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 28. Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The DE is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$, which has roots $-2, -2$. Thus, the general solution is $y(x) = y_p(x) + (C_1 + C_2 x)e^{-2x}$. It remains to find a particular solution y_p.
Since $(D-3)e^{3x} = 0$, we apply $(D-3)$ to the DE to get the **homogeneous** DE $(D-3)(D+2)^2 y = 0$.
Its general solution is $(C_1 + C_2 x)e^{-2x} + C_3 e^{3x}$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = Ae^{3x}$.
To determine the value of C, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ae^{3x} \overset{!}{=} e^{3x}$. Hence, $A = 1/25$. Therefore, the general solution to the original DE is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

Solution. (same, just shortened) In schematic form:

	homogeneous DE	inhomogeneous part
characteristic roots	$-2, -2$	3
solutions	e^{-2x}, xe^{-2x}	e^{3x}

This tells us that there exists a particular solution of the form $y_p = Ae^{3x}$. Then the general solution is

$$y = y_p + C_1 e^{-2x} + C_2 x e^{-2x}.$$

So far, we didn't need to do any calculations (besides determining the roots)! However, we still need to determine the value of A (by plugging into the DE as above), namely $A = \frac{1}{25}$. For this reason, this approach is often called the **method of undetermined coefficients**.

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We found the following recipe for solving nonhomogeneous linear DEs with constant coefficients:

That approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

(method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Determine the characteristic roots of the homogeneous DE and corresponding solutions.
- Find the roots of $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
  Let $y_{p,1}, y_{p,2}, \ldots$ be the additional solutions (when the roots are added to those of the homogeneous DE).

Then there exist (unique) C_i so that

$$y_p = C_1 y_{p,1} + C_2 y_{p,2} + \ldots$$

To find the values C_i, we need to plug y_p into the original DE.

Why? To see that this approach works, note that applying $q(D)$ to both sides of the inhomogeneous DE $p(D)y = f(x)$ results in $q(D)p(D)y = 0$ which is homogeneous. We already know that the solutions to the homogeneous DE can be added to any particular solution y_p. Therefore, we can focus only on the additional solutions coming from the roots of $q(D)$.

For which $f(x)$ does this work? By Theorem 20, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j e^{rx}$ (which includes $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$).

Example 29. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The homogeneous DE is $y'' + 4y' + 4y = 0$ (note that $D^2 + 4D + 4 = (D+2)^2$) and the inhomogeneous part is $7e^{-2x}$.

	homogeneous DE	inhomogeneous part
characteristic roots	$-2, -2$	-2
solutions	$e^{-2x}, x e^{-2x}$	$x^2 e^{-2x}$

This tells us that there exists a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $C = \frac{7}{2}$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. Hence the general solution is

$$y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}.$$

Example 30. Consider the DE $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution using our results from Examples 28 and 29.

(c) Determine the general solution.

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Solution.

(a) Note that $D^2+4D+4=(D+2)^2$.

	homogeneous DE	inhomogeneous part
characteristic roots	$-2, -2$	$3, -2$
solutions	e^{-2x}, xe^{-2x}	e^{3x}, x^2e^{-2x}

Hence, there has to be a particular solution of the form $y_p = Ae^{3x} + Bx^2e^{-2x}$.

To find the (unique) values of A and B, we can plug into the DE. Alternatively, we can break the problem into two pieces as illustrated in the next part.

(b) Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2+4D+4$. In Example 28 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 29 we found that $y_2 = \frac{7}{2}x^2e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1+By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution y_p of our DE, we need $A=2$ and $7B=-5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x}$.

Comment. Of course, if we hadn't previously solved Examples 28 and 29, we could have plugged the result from the first part into the DE to determine the coefficients A and B. On the other hand, breaking the inhomogeneous part ($2e^{3x}-5e^{-2x}$) up into pieces (here, e^{3x} and e^{-2x}) can help keep things organized, especially when working by hand.

(c) The general solution is $\frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x} + (C_1+C_2x)e^{2x}$.

Example 31. Consider the DE $y'' - 2y' + y = 5\sin(3x)$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution.

(c) Determine the general solution.

Solution. Note that $D^2-2D+1=(D-1)^2$.

	homogeneous DE	inhomogeneous part
characteristic roots	$1, 1$	$\pm 3i$
solutions	e^x, xe^x	$\cos(3x), \sin(3x)$

(a) This tells us that there exists a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

(b) To find the values of A and B, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A-6B)\cos(3x) + (6A-8B)\sin(3x) \overset{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A-6B=0$ and $6A-8B=5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

(c) The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1+C_2x)e^x$.

Example 32. Consider the DE $y'' - 2y' + y = 5e^{2x}\sin(3x) + 7xe^x$. What is the simplest form (with undetermined coefficients) of a particular solution?

Solution. Since $D^2-2D+1=(D-1)^2$, the characteristic roots are $1, 1$. The roots for the inhomogeneous part are $2\pm 3i, 1, 1$. Hence, there has to be a particular solution of the form $y_p = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x) + Cx^2e^x + Dx^3e^x$.

(We can then plug into the DE to determine the (unique) values of the coefficients A, B, C, D.)

Example 33. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2 x)\cos(x) + (C_3 + C_4 x)\sin(x)$.

Continuing to find a particular solution. To find the value of the C_j's, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4 x)\cos(x) + (C_4 - C_1 - C_2 x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2 x)\cos(x) + (-2C_2 - C_3 - C_4 x)\sin(x)$

$$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x) + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x).$$

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.