A crash course in linear algebra

Example 1. A typical 2×3 matrix is $\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{bmatrix}$.

It is composed of column vectors like $\begin{bmatrix} 2\\ 5 \end{bmatrix}$ and row vectors like $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

Matrices (and vectors) of the same dimensions can be added and multiplied by a scalar:

For instance, $\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2\\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5\\ 6 & 8 & 5 \end{bmatrix}$ or $3\cdot\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9\\ 12 & 15 & 18 \end{bmatrix}$.

Remark. More generally, a **vector space** is an abstraction of a collection of objects that can be added and scaled: numbers, lists of numbers (like the above row and column vectors), arrays of numbers (like the above matrices), arrows, functions, polynomials, differential operators, solutions to homogeneous linear differential equations, ...

Example 2. The **transpose** A^T of A is obtained by interchanging roles of rows and columns.

For instance. $\begin{bmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4\\ 2 & 5\\ 3 & 6 \end{bmatrix}$

Example 3. Matrices of appropriate dimensions can also be **multiplied**.

This is based on the multiplication $\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix} = ax + by + cz$ of row and column vectors.

For instance. $\begin{bmatrix} 1 & -1 & 1\\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0\\ -1 & 1\\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -3\\ 7 & -5 \end{bmatrix}$

In general, we can multiply a $m\times n$ matrix A with a $n\times r$ matrix B to get a $m\times r$ matrix AB.

Its entry in row i and column j is defined to be $(AB)_{ij} = (\text{row } i \text{ of } A)\begin{bmatrix} \text{column}\\ j\\ \text{of } B \end{bmatrix}$.

Comment. One way to think about the multiplication $A\boldsymbol{x}$ is that the resulting vector is a linear combination of the columns of A with coefficients from $\boldsymbol{x}$. Similarly, we can think of $\boldsymbol{x}^T A$ as a combination of the rows of A.

Some nice properties of matrix multiplication are:

- There is an $n\times n$ identity matrix I (all entries are zero except the diagonal ones which are 1). It satisfies $AI = A$ and $IA = A$.
- The associative law $A(BC) = (AB)C$ holds. Hence, we can write ABC without ambiguity.
- The distributive laws including $A(B+C) = AB + AC$ hold.

Example 4. $\begin{bmatrix} 2 & 0\\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2\\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2\\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0\\ 0 & 1 \end{bmatrix}$, so we have no commutative law.

Example 5. $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Hence, the two matrices on the left are inverses of each other: $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}$.

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The **inverse** A^{-1} of a matrix A is characterized by $A^{-1}A = I$ and $AA^{-1} = I$.

Example 6. The following formula immediately gives us the inverse of a 2×2 matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{provided that } ad-bc \neq 0$$

Let's check that! $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{bmatrix} = I_2$

In particular, a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad-bc \neq 0$.

Recall that this is the **determinant**: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$.

$$\det(A) = 0 \iff A \text{ is not invertible}$$

Example 7. The system $\begin{aligned} 7x_1 - 2x_2 &= 3 \\ 2x_1 + x_2 &= 5 \end{aligned}$ is equivalent to $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Solve it.

Solution. Multiplying (from the left!) by $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 13 \\ 29 \end{bmatrix}$, which gives the solution of the original equations.

Example 8. (homework) Solve the system $\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= -1 \end{aligned}$ (using a matrix inverse).

Solution. The equations are equivalent to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Multiplying by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Example 9. (homework) Solve the system $\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 2 \end{aligned}$ (using a matrix inverse).

Solution. The equations are equivalent to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Multiplying by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$.

Comment. In hindsight, can you see this solution by staring at the equations?

Comment. Note how we can reuse the matrix inverse from the previous example.

The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned} \det(A) \neq 0 &\iff A \text{ is invertible} \\ &\iff A\boldsymbol{x} = \boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ for all } \boldsymbol{b} \\ &\iff A\boldsymbol{x} = 0 \text{ is only solved by } \boldsymbol{x} = 0 \end{aligned}$$

Example 10. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$, which appeared in the formula for the inverse.

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