Midterm #1

MATH 332 — Differential Equations II
Friday, Feb 17, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 26 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (1+4+1 points) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = -2$, $a_1 = 6$.

(a) The next two terms are $a_2 =$ ☐ and $a_3 =$ ☐.

(b) A Binet-like formula for a_n is $a_n =$ ☐, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$ ☐.

Solution.

(a) $a_2 = 6$, $a_3 = 30$

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 2N - 3$ has roots $3, -1$.

Hence, $a_n = \alpha_1 3^n + \alpha_2 (-1)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = -2$, $a_1 = 3\alpha_1 - \alpha_2 = 6$.

Solving, we find $\alpha_1 = 1$ and $\alpha_2 = -3$ so that, in conclusion, $a_n = 3^n - 3(-1)^n$.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$.

Problem 2. (3 points) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 3y_1' - 5y_2 \\ y_2'' &= y_1' - y_2' + 3y_1 \end{aligned} \qquad y_1(0) = -2,\ y_1'(0) = 1,\ y_2(0) = 0,\ y_2'(0) = 3$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 3 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

Problem 3. (8 points) Fill in the blanks. None of the problems should require any computation!

(a) Consider a homogeneous linear differential equation with constant real coefficients which has order 4. Suppose $y(x) = 4x^2 e^{-x} + 7$ is a solution. Write down the general solution.

Armin Straub
straub@southalabama.edu

(b) Determine a (homogeneous linear) recurrence equation satisfied by $a_n = (n+2)3^n - 7$.

You can use the operator N to write the recurrence. No need to simplify, any form is acceptable.

(c) If $e^{Mx} = \begin{bmatrix} 2e^{2x} - e^x & e^{2x} - e^x \\ -2e^{2x} + 2e^x & -e^{2x} + 2e^x \end{bmatrix}$, then $M^n =$ ______.

(d) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + 5y = 3e^{2x} + 5x$. Find a homogeneous linear differential equation which y_p solves.

You can use the operator D to write the DE. No need to simplify, any form is acceptable.

Solution.

(a) $y(x) = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4$.

Explanation. $y(x) = 4x^2 e^{-x} + 7$ is a solution of $p(D)y = 0$ if and only if $-1, -1, -1, 0$ are roots of the characteristic polynomial $p(D)$. Since the order of the DE is 4, there can be no further roots. Hence, the general solution of this DE is $y(x) = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4$.

(b) $(N-3)^2(N-1)a_n = 0$

Explanation. $a_n = (n+2)3^n - 7$ is a solution of $p(N)a_n = 0$ if and only if 3 (repeated two times) and 1 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-3)^2(N-1)$.

[Since, $(N-3)^2(N-1) = N^3 - 7N^2 + 15N - 9$, the recurrence in explicit form is $a_{n+3} = 7a_{n+2} - 15a_{n+1} + 9a_n$.]

(c) $M^n = \begin{bmatrix} 2 \cdot 2^n - 1 & 2^n - 1 \\ -2 \cdot 2^n + 2 & -2^n + 2 \end{bmatrix}$

(d) $(D-2)D^2(D^2+5)y = 0$

Explanation. Since y_p solves the inhomogeneous DE, we have $(D^2+5)y_p = 3e^{2x} + 5x$. The right-hand side $3e^{2x} + 5x$ is a solution of $p(D)y = 0$ if and only if $2, 0, 0$ are roots of the characteristic polynomial $p(D)$. In particular, $(D-2)D^2(3e^{2x} + 5) = 0$. Combined, we find that $(D-2)D^2(D^2+5)y_p = 0$.

Problem 4. **(9 points)** Let $M = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix}$.

(a) Compute e^{Mx}.

(b) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 5-\lambda & 4 \\ 8 & 1-\lambda \end{bmatrix}\right) = (5-\lambda)(1-\lambda) - 32 = \lambda^2 - 6\lambda - 27 = (\lambda+3)(\lambda-9)$

Hence, the eigenvalues are $\lambda = -3$ and $\lambda = 9$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = -3$, we need to solve $\begin{bmatrix} 8 & 4 \\ 8 & 4 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = -3$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 9$, we need to solve $\begin{bmatrix} -4 & 4 \\ 8 & -8 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 9$.

Hence, a fundamental matrix solution is $\Phi = \begin{bmatrix} -e^{-3x} & e^{9x} \\ 2e^{-3x} & e^{9x} \end{bmatrix}$.

Note that $\Phi(0) = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} -e^{-3x} & e^{9x} \\ 2e^{-3x} & e^{9x} \end{bmatrix}\frac{1}{3}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{-3x} + 2e^{9x} & -e^{-3x} + e^{9x} \\ -2e^{-3x} + 2e^{9x} & 2e^{-3x} + e^{9x} \end{bmatrix}.$$

(b) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{-3x} + 2e^{9x} & -e^{-3x} + e^{9x} \\ -2e^{-3x} + 2e^{9x} & 2e^{-3x} + e^{9x} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -e^{-3x} + e^{9x} \\ 2e^{-3x} + e^{9x} \end{bmatrix}$.

(extra scratch paper)