Midterm #1 – Practice

MATH 332 — Differential Equations II
Midterm: Friday, Feb 17, 2023

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1.

(a) Find the general solution to $y^{(5)} - 4y^{(4)} + 5y''' - 2y'' = 0$.

(b) Find the general solution to $y''' - y = e^x + 7$.

(c) Solve $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

(d) Find the general solution to $y'' - 4y' + 4y = 3e^{2x}$.

(e) Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose $y(x) = x^2 e^{2x} \cos(x)$ is a solution. Write down the general solution.

(f) Write down a homogeneous linear differential equation satisfied by $y(x) = 1 - 5x^2 e^{-2x}$.

(g) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + xy = e^x$. Find a homogeneous linear differential equation which y_p solves. *Hint:* Do not attempt to solve the DE.

Solution.

(a) The characteristic polynomial $p(D) = D^5 - 4D^4 + 5D^3 - 2D^2 = D^2(D-1)^2(D-2)$ has roots $0, 0, 1, 1, 2$.

Hence, the general solution is $y(x) = c_1 + c_2 x + (c_3 + c_4 x)e^x + c_5 e^{2x}$.

(b) The characteristic polynomial $p(D) = D^3 - 1$ of the associated homogeneous DE has roots 1 and $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. These are the "old" roots.

The "new" roots coming from $e^x + 7$ are $0, 1$. Hence, there has to be a particular solution of the form $y_p = Axe^x + B$. To find the values of A, B, we plug into the DE.

$y_p' = A(x+1)e^x$, $y_p'' = A(x+2)e^x$, $y_p''' = A(x+3)e^x$

$y_p''' - y_p = 3Ae^x - B \overset{!}{=} e^x + 7$

Consequently, $A = \frac{1}{3}$, $B = -7$.

Hence, the general solution is $y(x) = -7 + (c_1 + \frac{1}{3}x)e^x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-x/2} \sin\left(\frac{\sqrt{3}}{2}x\right)$.

(c) The characteristic polynomial $p(D) = D^2 + 2D + 1$ of the associated homogeneous DE has roots $-1, -1$. These are the "old" roots.

The "new" roots coming from $2e^{2x} + e^{-x}$ are $-1, 2$. Hence, there has to be a particular solution of the form $y_p = Ae^{2x} + Bx^2 e^{-x}$. To find the values of A, B, we plug into the DE.

$y_p' = 2Ae^{2x} + B(2x - x^2)e^{-x}$, $y_p'' = 4Ae^{2x} + B(2 - 4x + x^2)e^{-x}$

$y_p'' + 2y_p' + y_p = 9Ae^{2x} + 2Be^{-x} \overset{!}{=} 2e^{2x} + e^{-x}$

Consequently, $A = \frac{2}{9}$, $B = \frac{1}{2}$.

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Hence, the general solution is $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2 e^{-x} + c_1 e^{-x} + c_2 x e^{-x}$. Now, we use the initial values to find the values for c_1 and c_2:

$y(0) = \frac{2}{9} + c_1 \overset{!}{=} -1$, so that $c_1 = -\frac{11}{9}$.

$y'(0) = \left[\frac{4}{9}e^{2x} + \left(x - \frac{1}{2}x^2\right)e^{-x} + \frac{11}{9}e^{-x} + c_2(1-x)e^{-x}\right]_{x=0} = \frac{5}{3} + c_2 \overset{!}{=} 2$, so that $c_2 = \frac{1}{3}$.

In conclusion, the unique solution to the IVP is $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2 e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}x e^{-x}$.

(d) The characteristic polynomial $p(D) = D^2 - 4D - 4$ of the associated homogeneous DE has "old" roots $2, 2$.

The "new" roots coming from $3e^{2x}$ are 2. Hence, there has to be a particular solution of the form $y_p = Ax^2 e^{2x}$. To find the value of A, we plug into the DE.

$y_p' = 2A(x + x^2)e^{2x}, \quad y_p'' = 2A(1 + 4x + 2x^2)e^{2x}$

$y_p'' - 4y_p' + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \overset{!}{=} 3e^{2x}$. Consequently, $A = \frac{3}{2}$.

Hence, the general solution is $\left(c_1 + c_2 x + \frac{3}{2}x^2\right)e^{2x}$.

(e) $y(x) = x^2 e^{2x}\cos(x)$ is a solution of $p(D)y = 0$ if and only if $2 \pm i$ are three times repeated roots of the characteristic polynomial $p(D)$. Since the order of the DE is 6, there can be no further roots.

The general solution of this DE is $y(x) = (c_1 + c_2 x + c_3 x^2)e^{2x}\cos(x) + (c_4 + c_5 x + c_6 x^2)e^{2x}\sin(x)$.

(f) $y(x) = 1 - 5x^2 e^{-2x}$ is a solution of $p(D)y = 0$ if and only if $-2, -2, -2, 0$ are roots of the characteristic polynomial $p(D)$. Hence, the simplest DE is obtained from $p(D) = D(D+2)^3 = D^4 + 6D^3 + 12D^2 + 8D$.

The corresponding recurrence is $y^{(4)} + 6y''' + 12y'' + 8y' = 0$.

(g) To kill e^x, we apply $D - 1$ to both sides of the DE $y'' + xy = e^x$.

The result is the homogeneous linear DE $y''' - y'' + xy' + (1-x)y = 0$.

Comment. If we are comfortable computing with operators, we can apply the relation $Dx = xD + 1$, to $(D-1)(D^2 + x) = D^3 - D^2 + Dx - x = D^3 - D^2 + xD + 1 - x$ to reach the same conclusion.

Problem 2.

(a) Write down a (homogeneous linear) recurrence equation satisfied by $a_n = 3^n - 2^n$.

(b) Write down a (homogeneous linear) recurrence equation satisfied by $a_n = n^2 3^n - 2^n$.

Solution.

(a) $a_n = 3^n - 2^n$ is a solution of $p(N)a_n = 0$ if and only if both 3 and 2 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-2)(N-3) = N^2 - 5N + 6$.

The corresponding recurrence is $a_{n+2} = 5a_{n+1} - 6a_n$.

(b) $a_n = n^2 3^n - 2^n$ is a solution of $p(N)a_n = 0$ if and only if 3 (repeated three times) and 2 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-2)(N-3)^3$.

The corresponding recurrence is $(N-2)(N-3)^3 a_n = 0$.

[Spelled out, this is $a_{n+4} = 11a_{n+3} - 45a_{n+2} + 81a_{n+1} - 54a_n$.]

Problem 3. Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 3$, $a_1 = -1$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = 17$, $a_3 = 11$

(b) The recursion can be written as $p(N) a_n = 0$ where $p(N) = N^2 - N - 6$ has roots $3, -2$.

Hence, $a_n = \alpha_1 3^n + \alpha_2 (-2)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 3$, $a_1 = 3\alpha_1 - 2\alpha_2 = -1$.

Solving, we find $\alpha_1 = 1$ and $\alpha_2 = 2$ so that, in conclusion, $a_n = 3^n + 2 \cdot (-2)^n$.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$.

Problem 4. Let $M = \begin{bmatrix} 1 & 4 \\ 6 & -1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M \boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M \boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 6 & -1-\lambda \end{bmatrix} \right) = (1-\lambda)(-1-\lambda) - 24 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = -5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 5$, we need to solve $\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = -5$, we need to solve $\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda = -5$.

Hence, the general solution is $C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} 5^n + C_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} (-5)^n$.

(b) The corresponding fundamental matrix solution is $\Phi_n = \begin{bmatrix} 5^n & -2(-5)^n \\ 5^n & 3(-5)^n \end{bmatrix}$.

(c) Note that $\Phi_0 = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, so that $\Phi_0^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$. It follows that

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 5^n & -2(-5)^n \\ 5^n & 3(-5)^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \cdot 5^n + 2(-5)^n & 2 \cdot 5^n - 2(-5)^n \\ 3 \cdot 5^n - 3(-5)^n & 2 \cdot 5^n + 3(-5)^n \end{bmatrix}.$$

Problem 5.

(a) Write the differential equation $y''' + 7y'' - 3y' + y = 0$ as a system of (first-order) differential equations.

(b) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 3y_1' + 2y_2' - 5y_1 \\ y_2'' &= y_1' - y_2' + 3y_2 \end{aligned} \qquad y_1(0) = 1, \ y_1'(0) = -2, \ y_2(0) = 3, \ y_2'(0) = 0$$

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Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution.

(a) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' + 7y'' - 3y' + y = 0$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_1 + 3y_2 - 7y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & -7 \end{bmatrix} \boldsymbol{y}$.

(b) Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & 3 & 2 \\ 0 & 3 & 1 & -1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}.$$

Problem 6. Let $M = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 11-\lambda & -2 \\ 3 & 4-\lambda \end{bmatrix}\right) = (11-\lambda)(4-\lambda) + 6 = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10)$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = 10$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 5$, we need to solve $\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda = 5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 10$, we need to solve $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 10$.

Hence, the general solution is $C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{5x} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{10x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} e^{5x} & 2e^{10x} \\ 3e^{5x} & e^{10x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} e^{5x} & 2e^{10x} \\ 3e^{5x} & e^{10x} \end{bmatrix} \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -e^{5x} + 6e^{10x} & 2e^{5x} - 2e^{10x} \\ -3e^{5x} + 3e^{10x} & 6e^{5x} - e^{10x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -e^{5x} + 6e^{10x} & 2e^{5x} - 2e^{10x} \\ -3e^{5x} + 3e^{10x} & 6e^{5x} - e^{10x} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3e^{5x} + 8e^{10x} \\ -9e^{5x} + 4e^{10x} \end{bmatrix}$.