Sketch of Lecture 1

Mon, 1/9/2023

A crash course in linear algebra

Example 1. A typical 2×3 matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

It is composed of column vectors like $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and row vectors like $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

Matrices (and vectors) of the same dimensions can be added and multiplied by a scalar:

For instance, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 6 & 8 & 5 \end{bmatrix}$ or $3\cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$.

Remark. More generally, a **vector space** is an abstraction of a collection of objects that can be added and scaled: numbers, lists of numbers (like the above row and column vectors), arrays of numbers (like the above matrices), arrows, functions, polynomials, differential operators, solutions to homogeneous linear differential equations, $\ldots$

Example 2. The **transpose** A^T of A is obtained by interchanging roles of rows and columns.

For instance. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Example 3. Matrices of appropriate dimensions can also be **multiplied**.

This is based on the multiplication $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$ of row and column vectors.

For instance. $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 7 & -5 \end{bmatrix}$

In general, we can multiply a $m\times n$ matrix A with a $n\times r$ matrix B to get a $m\times r$ matrix AB.

Its entry in row i and column j is defined to be $(AB)_{ij} = (\text{row } i \text{ of } A) \begin{bmatrix} \text{column} \\ j \\ \text{of } B \end{bmatrix}$.

Comment. One way to think about the multiplication $A\boldsymbol{x}$ is that the resulting vector is a linear combination of the columns of A with coefficients from $\boldsymbol{x}$. Similarly, we can think of $\boldsymbol{x}^T A$ as a combination of the rows of A.

Some nice properties of matrix multiplication are:

- There is an $n\times n$ identity matrix I (all entries are zero except the diagonal ones which are 1). It satisfies $AI = A$ and $IA = A$.
- The associative law $A(BC) = (AB)C$ holds. Hence, we can write ABC without ambiguity.
- The distributive laws including $A(B + C) = AB + AC$ hold.

Example 4. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, so we have no commutative law.

Example 5. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Hence, the two matrices on the left are inverses of each other: $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

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The **inverse** A^{-1} of a matrix A is characterized by $A^{-1}A = I$ and $AA^{-1} = I$.

Example 6. The following formula immediately gives us the inverse of a 2×2 matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{provided that } ad - bc \neq 0$$

Let's check that! $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = I_2$

In particular, a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Longleftrightarrow$ $ad - bc \neq 0$.

Recall that this is the **determinant**: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

$$\det(A) = 0 \quad \Longleftrightarrow \quad A \text{ is not invertible}$$

Example 7. The system $\begin{aligned} 7x_1 - 2x_2 &= 3 \\ 2x_1 + x_2 &= 4 \end{aligned}$ is equivalent to $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Solve it.

Solution. Multiplying (from the left!) by $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which gives the solution of the original equations.

The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned} \det(A) \neq 0 \quad &\Longleftrightarrow \quad A \text{ is invertible} \\ &\Longleftrightarrow \quad A\boldsymbol{x} = \boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ for all } \boldsymbol{b} \\ &\Longleftrightarrow \quad A\boldsymbol{x} = 0 \text{ is only solved by } \boldsymbol{x} = 0 \end{aligned}$$

Example 8. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$, which appeared in the formula for the inverse.

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Example 9. (review) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Review: Examples of differential equations we can solve

Let's start with one of the simplest (and most fundamental) differential equations (DE). It is **first-order** (only a first derivative) and **linear** (with constant coefficients).

Example 10. Solve $y' = 3y$.

Solution. $y(x) = Ce^{3x}$

Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 11. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **non-linear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 12. Solve $y' = xy^2$.

Solution. This DE is separable: $\frac{1}{y^2}\mathrm{d}y = x\,\mathrm{d}x$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$.

[Here, $D = -2C$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Note. Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $D \to \infty$.]

Check. Compute y' and verify that the DE is indeed satisfied.

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Review: Linear first-order DEs

The most general first-order linear DE is $y' = a(x)y + f(x)$.

We will recall next time that we can always solve it.

The corresponding **homogeneous** linear DE is $y' = a(x)y$.

Important comment. Write $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then we can write $y' - a(x)y = f(x)$ as $Ly = f(x)$ where $L = D - a(x)$. The corresponding homogeneous DE is simply $Ly = 0$.

Solving linear first-order DEs using variation of constants

The following DE is linear and first-order (but not with constant coefficients).

Example 13. Solve $y' = x^2 y$.

Solution. This DE is separable as well: $\frac{1}{y}\mathrm{d}y = x^2\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = \frac{1}{3}x^3 + A$, so that $|y| = e^{\frac{1}{3}x^2 + A}$. Since the RHS is never zero, we must have either $y = e^{\frac{1}{3}x^2 + A}$ or $y = -e^{\frac{1}{3}x^2 + A}$.

Hence $y = \pm e^A e^{\frac{1}{3}x^3} = Ce^{\frac{1}{3}x^3}$ (with $C = \pm e^A$). Note that $C = 0$ corresponds to the singular solution $y = 0$.

In summary, the general solution is $y = Ce^{\frac{1}{3}x^3}$ (with C any real number).

Check. Compute y' and verify that the DE is indeed satisfied.

As in the previous example, we can immediately solve any **homogeneous** linear first-order DE:

Example 14. Solve $y' = a(x)y$.

Solution. Proceeding as in the previous example, we find $y(x) = Ce^{\int a(x)\mathrm{d}x}$.

Check. Compute y' and verify that the DE is indeed satisfied.

Recall that, to find the general solution of the inhomogeneous DE $y' = a(x)y + f(x)$, we only need to find a particular solution y_p.

Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE $y' = a(x)y$.

Theorem 15. (variation of constants) $y' = a(x)y + f(x)$ has the particular solution

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x)y$.

Proof. $y_p'(x) = y_h'(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + y_h(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\int \frac{f(x)}{y_h(x)}\mathrm{d}x}_{\frac{f(x)}{y_h(x)}} = a(x)y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + f(x) = a(x)y + f(x)$ □

Comment. Note that the formula for $y_p(x)$ gives the general solution if we let $\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ be the general antiderivative. (Think about the effect of the constant of integration!)

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Recall. The formula for $y_p(x)$ can be found using **variation of constants** (sometimes called variation of parameters): that is, we look for solutions of the form $y(x) = c(x) y_h(x)$.

If we plug $y(x) = c(x) y_h(x)$ into the DE, we find $c' y_h + c y_h' = a c y_h + f$. Since $y_h' = a y_h$, this simplifies to $c' y_h = f$ or, equivalently, $c' = \frac{f}{y_h}$. Hence, $c(x) = \int \frac{f(x)}{y_h(x)} \mathrm{d}x$, which is the formula in the theorem.

Example 16. Solve $x^2 y' = 1 - x y + 2x$, $y(1) = 3$.

Solution. Write as $\frac{\mathrm{d}y}{\mathrm{d}x} = a(x) y + f(x)$ with $a(x) = -\frac{1}{x}$ and $f(x) = \frac{1}{x^2} + \frac{2}{x}$.

$y_h(x) = e^{\int a(x) \mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$. (Why can we write $\ln x$ instead of $\ln|x|$?!) Hence:

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)} \mathrm{d}x = \frac{1}{x} \int \left(\frac{1}{x} + 2\right) \mathrm{d}x = \frac{\ln x + 2x + C}{x}$$

Using $y(1) = 3$, we find $C = 1$. In summary, the solution is $y = \frac{\ln(x) + 2x + 1}{x}$.

Comment. Observe how the general solution (with parameter C) is indeed obtained from any particular solution (say, $\frac{\ln x + 2x}{x}$) plus the general solution to the homogeneous equation, which is $\frac{C}{x}$.

Review: Linear DEs

A linear DE of order n is of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $L y = f(x)$ with $L = D^n + p_{n-1}(x) D^{n-1} + \ldots + p_1(x) D + p_0(x)$.
  Comment. L is called a (linear) differential operator.
- The inclusion of the $f(x)$ term makes $L y = f(x)$ an **inhomogeneous** linear DE.
- $L y = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of the homogeneous DE)** There are n solutions $y_1, y_2, \ldots, y_n$, such that every solution is of the form $C_1 y_1 + \ldots + C_n y_n$. [These n solutions necessarily are **independent**.]
- To find the general solution of the inhomogeneous DE, we only need to find a single solution y_p (called a **particular solution**). Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE.

Homogeneous linear DEs with constant coefficients

Example 17. Find the general solution to $y'' - y' - 2y = 0$.

Solution. We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2 e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operators) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2) y = 0$.
Note that $D^2 - D - 2 = (D-2)(D+1)$ is the **characteristic polynomial**.
It follows that we get solutions to $(D-2)(D+1) y = 0$ from $(D-2) y = 0$ and $(D+1) y = 0$.
$(D-2) y = 0$ is solved by $y_1 = e^{2x}$, and $(D+1) y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.

Example 18. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From the previous example, we know that $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
To match the initial conditions, we need to solve $C_1 + C_2 = 4$, $2C_1 - C_2 = 5$. We find $C_1 = 3$, $C_2 = 1$.
Hence the solution is $y(x) = 3e^{2x} + e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D) y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $L y = 0$ with $L = D^2 - D - 2$.

Example 19. Find the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

Solution. This DE is of the form $p(D)\, y = 0$ with characteristic polynomial $p(D) = D^3 + 7D^2 + 14D + 8$.
The characteristic polynomial factors as $p(D) = (D+1)(D+2)(D+4)$. (Don't worry! You won't be asked to factor cubic polynomials by hand.)
Hence, we found the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. That's enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x) = C_1\, e^{-x} + C_2\, e^{-2x} + C_3\, e^{-4x}$.

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This approach applies to any homogeneous linear DE with constant coefficients!

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following result provides the missing solutions.

Theorem 20. Consider the homogeneous linear DE with constant coefficients $p(D)y = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. Likewise, if $r = a \pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) solutions of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j = 0, 1, \ldots, k-1$.

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D) = q(D)\,(D-r)^k$.

We need to find k solutions to the simpler DE $(D-r)^k y = 0$.

It is natural to look for solutions of the form $y = c(x)e^{rx}$.

[We know that $c(x) = 1$ provides a solution. Note that this is the same idea as for variation of constants.]

Note that $(D-r)[c(x)e^{rx}] = (c'(x)e^{rx} + c(x)re^{rx}) - rc(x)e^{rx} = c'(x)e^{rx}$.

Repeating, we get $(D-r)^2[c(x)e^{rx}] = (D-r)[c'(x)e^{rx}] = c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}] = c^{(k)}(x)e^{rx}$.

In particular, $(D-r)^k y = 0$ is solved by $y = c(x)e^{rx}$ if and only if $c^{(k)}(x) = 0$.

The DE $c^{(k)}(x) = 0$ is clearly solved by x^j for $j = 0, 1, \ldots, k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 21. Find the general solution of $y''' = 0$.

Solution. We know from Calculus that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Solution. The characteristic polynomial $p(D) = D^3$ has roots $0, 0, 0$. By Theorem 20, we have the solutions $y(x) = x^j e^{0x} = x^j$ for $j = 0, 1, 2$, so that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Example 22. Find the general solution of $y''' - 3y' + 2y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D + 2 = (D-1)^2(D+2)$ has roots $1, 1, -2$.

By Theorem 20, the general solution is $y(x) = (C_1 + C_2 x)e^x + C_2 e^{-2x}$.

Example 23. **(review)** Find the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$. Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x)e^{-x}$.

Example 24. Find the general solution of $y'' + y = 0$.

Solution. The characteristic polynomials is $p(D) = D^2 + 1 = 0$ which has no solutions over the reals.
Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1\, e^{ix} + C_2\, e^{-ix}$.

Solution. On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x) = D_1 \cos(x) + D_2 \sin(x)$.

Important comment. That we have these two different representations is a consequence of **Euler's identity**

$$e^{ix} = \cos(x) + i \sin(x).$$

Note that $e^{-ix} = \cos(x) - i \sin(x)$.

On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 25. Find the general solution of $y'' - 4y' + 13y = 0$.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.
Hence, the general solution is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.
Note. $e^{(2+3i)x} = e^{2x}e^{3ix} = e^{2x}(\cos(3x) + i \sin(3x))$

Inhomogeneous linear DEs with constant coefficients

Example 26. Find the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.
Hence, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.
Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.
We get $D^2(D^2 + 4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.
It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$.
[Why?! Because we know that $C_3\cos(2x) + C_4\sin(2x)$ can be added to any particular solution.]
It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.
Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 27. Find the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. This is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$.
Hence, the general solution is $y(x) = y_p(x) + (C_1 + C_2 x)e^{-2x}$. It remains to find a particular solution y_p.
Note that $(D-3)e^{3x} = 0$. Hence, we apply $(D-3)$ to the DE to get $(D-3)(D+2)^2 y = 0$.
This homogeneous linear DE has general solution $(C_1 + C_2 x)e^{-2x} + C_3 e^{3x}$. We conclude that the original DE must have a particular solution of the form $y_p = C_3 e^{3x}$.
To determine the value of C_3, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)C_3 e^{3x} \overset{!}{=} e^{3x}$. Hence, $C_3 = 1/25$. In conclusion, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

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We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

Theorem 28. (method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
- Let $r_1, \ldots, r_n$ be the ("old") roots of the polynomial $p(D)$.
  Let $s_1, \ldots, s_m$ be the ("new") roots of the polynomial $q(D)$.
- It follows that y_p solves the **homogeneous** DE $q(D)\,p(D)y = 0$.
  The characteristic polynomial of this DE has roots $r_1, \ldots, r_n, s_1, \ldots, s_m$.
  Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).
  By plugging into $p(D)y_p = f(x)$, we find (unique) C_i so that $y_p = C_1 v_1 + \ldots + C_m v_m$.

Because of the final step, this approach is often called **method of undetermined coefficients**.

For which $f(x)$ does this work? By Theorem 20, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j\, e^{rx}$ (which includes $x^j\, e^{ax}\cos(bx)$ and $x^j\, e^{ax}\sin(bx)$).

Example 29. (again) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25} e^{3x}$.

Example 30. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \stackrel{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 31. Determine a particular solution of $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

Solution. Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. Instead of starting all over, recall that in Example 29 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 30 we found that $y_2 = \frac{7}{2}x^2 e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution y_p of our DE, we need $A = 2$ and $7B = -5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2 e^{-2x}$.

Example 32. (homework) Determine the general solution of $y'' - 2y' + y = 5\sin(3x)$.

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $\pm 3i$. Hence, there has to be a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

To find the values of A and B, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \stackrel{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2 x)e^x$.

Example 33. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The "old" roots are $-2, -2$. The "new" roots are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2 x)\cos(x) + (C_3 + C_4 x)\sin(x)$.

Continuing to find a particular solution. To find the value of the C_j's, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4 x)\cos(x) + (C_4 - C_1 - C_2 x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2 x)\cos(x) + (-2C_2 - C_3 - C_4 x)\sin(x)$

$$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x) + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \stackrel{!}{=} x\cos(x).$$

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

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Example 34. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

Solution. The "old" roots are $-2, -2$. The "new" roots are $3 \pm 2i, \pm i, \pm i$.
Hence, there has to be a particular solution of the form
$y_p = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x) + (C_3 + C_4 x)\cos(x) + (C_5 + C_6 x)\sin(x)$.

Continuing to find a particular solution. To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

Excursion: Euler's identity

Theorem 35. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 36. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x + y) = \ldots$ (that's what we actually did in class).

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

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Example 37. We have been factoring differential operators like $D^2+4D+4=(D+2)^2$.

Things become much more complicated when the coefficients are not constant!

For instance, the linear DE $y''+4y'+4xy=0$ can be written as $Ly=0$ with $L=D^2+4D+4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $xD\neq Dx$!!

Indeed, $(xD)f(x)=xf'(x)$ while $(Dx)f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)]=f(x)+xf'(x)=(1+xD)f(x)$.

This computation shows that, in fact, $Dx=xD+1$.

Review. Linear DEs are those that can be written as $Ly=f(x)$ where L is a linear differential operator: namely,

$$L=p_n(x)D^n+p_{n-1}(x)D^{n-1}+\ldots+p_1(x)D+p_0(x). \tag{1}$$

Recall that the operators xD and Dx are not the same: instead, $Dx=xD+1$.

We say that an operator of the form (1) is in **normal form**.

For instance. xD is in normal form, whereas Dx is not in normal form. It follows from the previous example tha the normal form of Dx is $xD+1$.

Example 38. Let $a=a(x)$ be some function.

(a) Write the operator Da in normal form [normal form means as in (1)].

(b) Write the operator D^2a in normal form.

Solution.

(a) $(Da)f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[a(x)\,f(x)]=a'(x)f(x)+a(x)f'(x)=(a'+aD)f(x)$
Hence, $Da=aD+a'$.

(b) $(D^2a)f(x)=\frac{\mathrm{d}^2}{\mathrm{d}x^2}[a(x)\,f(x)]=\frac{\mathrm{d}}{\mathrm{d}x}[a'(x)f(x)+a(x)f'(x)]=a''(x)f(x)+2a'(x)f'(x)+a(x)f''(x)$
$=(a''+2a'D+aD^2)f(x)$
Hence, $D^2a=aD^2+2a'D+a''$.

Example 39. Suppose that a and b depend on x. Expand $(D+a)(D+b)$ in normal form.

Solution. $(D+a)(D+b)=D^2+Db+aD+ab=D^2+(bD+b')+aD+ab=D^2+(a+b)D+ab+b'$

Comment. Of course, if b is a constant, then $b'=0$ and we just get the familiar expansion.

Comment. At this point, it is not surprising that, in general, $(D+a)(D+b)\neq(D+b)(D+a)$.

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Example 40. Suppose we want to factor $D^2 + pD + q$ as $(D+a)(D+b)$. [p, q, a, b depend on x]

(a) Spell out equations to find a and b.

(b) Find all factorizations of D^2. [An obvious one is $D^2 = D \cdot D$ but there are others!]

Solution.

(a) Matching coefficients with $(D+a)(D+b) = D^2 + (a+b)D + ab + b'$, we find that we need

$$p = a + b, \quad q = ab + b'.$$

Equivalently, $a = p - b$ and $q = (p-b)b + b'$. The latter is a nonlinear (!) DE for b. Once solved for b, we obtain a as $a = p - b$.

(b) This is the case $p = q = 0$. The DE for b becomes $b' = b^2$.
Because it is separable (show all details!), we find that $b(x) = \frac{1}{C - x}$ or $b(x) = 0$.

Since $a = -b$, we obtain the factorizations $D^2 = \left(D - \frac{1}{C-x}\right)\left(D + \frac{1}{C-x}\right)$ and $D^2 = D \cdot D$.

Our computations show that there are no further factorizations.

Comment. Note that this example illustrates that factorization of differential operators is not unique!
For instance, $D^2 = D \cdot D$ and $D^2 = \left(D + \frac{1}{x}\right) \cdot \left(D - \frac{1}{x}\right)$ (the case $C = 0$ above).

Comment. In general, the nonlinear DE for b does not have any polynomial or rational solution (or, in fact, any solution that can be expressed in terms of functions that we are familiar with).

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