Example 142. Find the unique solution $u(x, y)$ to:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 1 \\ u(x,2) &= 0 \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned} \quad \text{(BC)}$$

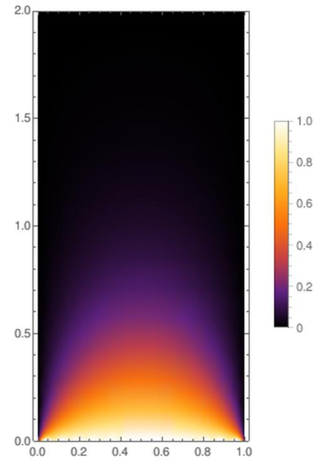
Solution. This is the special case of the previous example with $a = 1$, $b = 2$ and $f(x) = 1$ for $x \in (0, 1)$.

From Example 116, we know that $f(x)$ has the Fourier sine series

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi x), \quad x \in (0,1).$$

Hence,

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{1}{1 - e^{4\pi n}} \sin(\pi n x)(e^{\pi n y} - e^{\pi n (4-y)}).$$



Comment. The temperature at the center is $u(\frac{1}{2}, 1) \approx 0.0549$ (only the first term of the infinite sum suffices for this estimate; the first three terms suffice for 9 digits of accuracy).

Example 143. Find the unique solution $u(x, y)$ to:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{(PDE)} \\ u(x,0) &= 0 \\ u(x,2) &= 3 \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned} \quad \text{(BC)}$$

Solution. Instead of starting from scratch (homework exercise!), let us reuse our computations:
Let $v(x,y) = u(x, 2-y)$. Then $v_{xx} + v_{yy} = 0$, $v(x,0) = 3$, $v(x,2) = 0$, $v(0,y) = 0$, $v(1,y) = 0$.
Hence, it follows from the previous example that

$$v(x,y) = 3\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{1}{1 - e^{4\pi n}} \sin(\pi n x)(e^{\pi n y} - e^{\pi n (4-y)}).$$

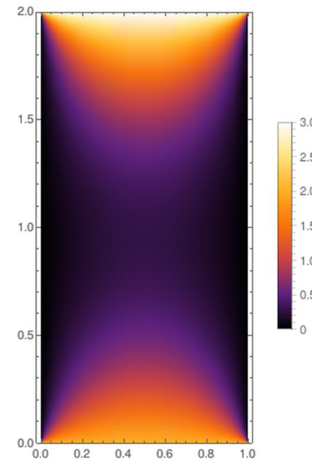
Consequently,

$$u(x,y) = v(x, 2-y) = 3\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{1}{1 - e^{4\pi n}} \sin(\pi n x)(e^{\pi n (2-y)} - e^{\pi n (2+y)}).$$

Armin Straub
straub@southalabama.edu

Example 144. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \\ u(x,0)=2, \quad u(x,2)&=3 \\ u(0,y)=0, \quad u(1,y)&=0 \end{aligned}$$



Solution. Note that $u(x,y)$ is a combination of the solutions to the previous two examples!

$$u(x,y)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\frac{\sin(\pi n x)}{1-e^{4\pi n}}[2(e^{\pi n y}-e^{-\pi n(y-4)})+3(e^{\pi n(2-y)}-e^{\pi n(2+y)})].$$

Example 145. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy} &= 0 && \text{(PDE)} \\ u(x,0) &= 4\sin(\pi x)-5\sin(3\pi x) \\ u(x,2) &= 0 && \text{(BC)} \\ u(0,y) &= 0 \\ u(3,y) &= 0 \end{aligned}$$

Solution.

- We look for solutions $u(x,y)=X(x)Y(y)$ (**separation of variables**).
  Plugging into (PDE), we get $X''(x)Y(y)+X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\text{const}$.
  We thus have $X''-\text{const}\,X=0$ and $Y''+\text{const}\,Y=0$.
- From the last three (BC), we get $X(0)=0$, $X(3)=0$, $Y(2)=0$.
- So X solves $X''+\lambda X=0$ (we choose $\lambda=-\text{const}$), $X(0)=0$, $X(3)=0$.
  From earlier (or do it!), we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x)=\sin\left(\frac{1}{3}\pi n x\right)$ corresponding to $\lambda=\left(\frac{1}{3}\pi n\right)^2$, $n=1,2,3\ldots$.
- On the other hand, Y solves $Y''-\lambda Y=0$, and hence $Y(y)=Ae^{\sqrt{\lambda}y}+Be^{-\sqrt{\lambda}y}$.
  The condition $Y(2)=0$ implies that $Ae^{2\sqrt{\lambda}}+Be^{-2\sqrt{\lambda}}=0$ so that $B=-Ae^{4\sqrt{\lambda}}$.
  Hence, $Y(y)=A(e^{\sqrt{\lambda}y}-e^{\sqrt{\lambda}(4-y)})=A\left(e^{\frac{1}{3}\pi n y}-e^{\frac{1}{3}\pi n(4-y)}\right)$.
- Taken together, we have the solutions $u_n(x,y)=\sin\left(\frac{1}{3}\pi n x\right)\left(e^{\frac{1}{3}\pi n y}-e^{\frac{1}{3}\pi n(4-y)}\right)$ solving (PDE)+(BC), with the exception of $u(x,0)=4\sin(\pi x)-5\sin(3\pi x)$.
- At $y=0$, $u_n(x,0)=\sin\left(\frac{1}{3}\pi n x\right)\left(1-e^{\frac{4}{3}\pi n}\right)$.
  In particular, $u_3(x,0)=\sin(\pi x)(1-e^{4\pi})$ and $u_9(x,0)=\sin(3\pi x)(1-e^{12\pi})$.
  Hence, $4\sin(\pi x)-5\sin(3\pi x)=\frac{4}{1-e^{4\pi}}u_3(x,0)-\frac{5}{1-e^{12\pi}}u_9(x,0)$. Therefore our overall solution is

$$\begin{aligned} u(x,y) &= \frac{4}{1-e^{4\pi}}u_3(x,y)-\frac{5}{1-e^{12\pi}}u_9(x,y) \\ &= \frac{4}{1-e^{4\pi}}\sin(\pi x)(e^{\pi y}-e^{\pi(4-y)})-\frac{5}{1-e^{12\pi}}\sin(3\pi x)(e^{3\pi y}-e^{3\pi(4-y)}). \end{aligned}$$

Comment. Of course, in general, our inhomogeneous (BC) will be a function $f(x)$ that is not such an obvious combination of our special solutions $u_n(x,0)$. In that case, we need to compute an appropriate Fourier expansion of $f(x)$ first (here, the Fourier sine series of $f(x)$).